MATH1853 Linear Algebra, Probability & Statistics
Lecture Notes

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Important: *This course has two parts: (1) Linear Algebra; and (2) Probability & Statistics. These notes only cover the second part.*

1 Complex Variables

Definition 1.1 (Complex Number)**.** The set of **complex numbers** is defined as

$$\mathbb{C} = \{a + b\mathrm{i} \mid a, b \in \mathbb{R}, \mathrm{i}^2 = -1\}$$

Further, for $z = a + b\mathrm{i} \in \mathbb{C}$, we refer to a as the **real part** of z, denoted $\mathrm{Re}(z)$, and b as the **imaginary part** of z, denoted $\mathrm{Im}(z)$.

Definition 1.2 (Complex Operations)**.** $\mathbb{C}$ is a field under the certain operations. For $z = a + b\mathrm{i}, w = c + d\mathrm{i} \in \mathbb{C}$, we define:

Name	Notation	Definition
Addition	$z + w$	$(a + c) + (b + d)\mathrm{i}$
Subtraction	$z - w$	$(a - c) + (b - d)\mathrm{i}$
Multiplication	zw	$(ac - bd) + (ad + bc)\mathrm{i}$
Division	$\frac{z}{w}$, $(w \neq 0)$	$\frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}\mathrm{i}$
Conjugate	$\overline{z}$	$a - b\mathrm{i}$
Modulus	$\lvert z\rvert$	$\sqrt{a^2 + b^2}$

Table 1: Operations on Complex Numbers

Theorem 1.3 (Triangle Inequality)**.** For any $z, w \in \mathbb{C}$, we have

$$|z + w| \leq |z| + |w|$$

Proof.
The proof is as follows:

$$\begin{aligned}
|z + w|^2 &= (z + w)\overline{(z + w)} \\
&= z\overline{z} + w\overline{w} + z\overline{w} + \overline{z}w \\
&= |z|^2 + |w|^2 + z\overline{w} + \overline{z\overline{w}} \\
&= |z|^2 + |w|^2 + 2\,\mathrm{Re}(z\overline{w}) \qquad (\text{by } \mathrm{Re}(z) = \frac{z + \overline{z}}{2}) \\
&\leq |z|^2 + |w|^2 + 2|z\overline{w}| \qquad (\sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2} \geq \mathrm{Re}(z)) \\
&= |z|^2 + |w|^2 + 2|z||w| \\
&= (|z| + |w|)^2
\end{aligned}$$

Q.E.D.

Theorem 1.4 (Properties of Conjugates and Moduli). For any $z, w \in \mathbb{C}$, we have

1. **properties of conjugates:**
   (a) $\overline{(z+w)} = \overline{z} + \overline{w}$
   (b) $\overline{(zw)} = \overline{z}\,\overline{w}$
   (c) $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$, $(w \neq 0)$
   (d) $\overline{(\overline{z})} = z$ (**involution**)
2. **properties of moduli:**
   (a) $|z| \geq 0$, and $|z| = 0$ if and only if $z = 0$
   (b) $|zw| = |z||w|$
   (c) $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$, $(w \neq 0)$
   (d) $|z|^n = |z^n|$

Definition 1.5 (Polar Form of a Complex Number). For any $z = a + bi \in \mathbb{C}$, we can express z in **polar form** as

$$z = r(\cos\theta + \mathrm{i}\sin\theta)$$

where $r = |z| = \sqrt{a^2 + b^2}$ is the **modulus** of z, and $\theta = \arg(z) = \arctan\frac{b}{a}$ is the **argument** of z.

Remark. In MATH1853, we limit θ to be the principal argument, i.e., $\arg(z) \in [0, 2\pi)$ or $(-\pi, \pi]$.

Definition 1.6 (Multiplication of Complex Numbers in Polar Form). Let $z_1, z_2 \in \mathbb{C}$, where $z_1 = r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)$. Then, the multiplication of z_1 and z_2 is given by

$$z_1 z_2 = r_1 r_2 \left[\cos(\theta_1 + \theta_2) + \mathrm{i}\sin(\theta_1 + \theta_2)\right]$$

Theorem 1.7 (De Moivre's Theorem). If $z = r(\cos\theta + \mathrm{i}\sin\theta) \in \mathbb{C}$, $n \in \mathbb{Z}$, $r > 0$, then

$$z^n = r^n \left[\cos(n\theta) + \mathrm{i}\sin(n\theta)\right]$$

Theorem 1.8 (Euler's Formula). The formula states:

$$e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$$

Corollary 1.9. The trigonometric functions can be expressed in terms of exponentials as follows:

$$\cos\theta = \frac{e^{\mathrm{i}\theta} + e^{-\mathrm{i}\theta}}{2}, \quad \sin\theta = \frac{e^{\mathrm{i}\theta} - e^{-\mathrm{i}\theta}}{2\mathrm{i}}$$

Definition 1.10 (Exponential Form of a Complex Number). For any $z = a + bi \in \mathbb{C}$, z has the **exponential form**:

$$z = re^{\mathrm{i}\theta}$$

where $r = |z| = \sqrt{a^2 + b^2}$ and $\theta = \arg(z) = \arctan\frac{b}{a}$.

Theorem 1.11 (Properties of the Exponential Form). For any $z, w \in \mathbb{C}$, we have the following properties:

Property	**Rule**
Product	$e^z e^w = e^{z+w}$
Quotient	$\frac{e^z}{e^w} = e^{z-w}$
Power	$(e^z)^n = e^{nz}$
De Moivre's	$(re^{\mathrm{i}\theta})^n = r^n e^{\mathrm{i}n\theta}$
Conjugage	$\overline{e^z} = e^{\overline{z}}$
Modulus	$\lvert e^z \rvert = e^{\mathrm{Re}(z)}$
Argument	$\arg(e^z) = \mathrm{Im}(z)$