MATH1853 Linear Algebra, Probability & Statistics
25/26 Semester 1 – Part II
Assignment 2 (Proposed Solutions)

Remarks: *These are the proposed solutions of the author. They shall by no means be considered as an official solution provided by the department. They may contain errors. You are advised to use with discretion.*

1. You are planning a photo session with your 7 best friends. In how many ways can your 7 friends stand in a line for the photo if you must be standing in between your two childhood friends immediately on your left and right?

Solution:

There are 2 ways to arrange your two childhood friends on your left and right. Then, treating you and your two childhood friends as an entity, there are 6 entities to arrange. Thus, there are 6! ways to arrange these 6 entities. Therefore, the total number of ways is $2 \times 6! = \boxed{1440}$.

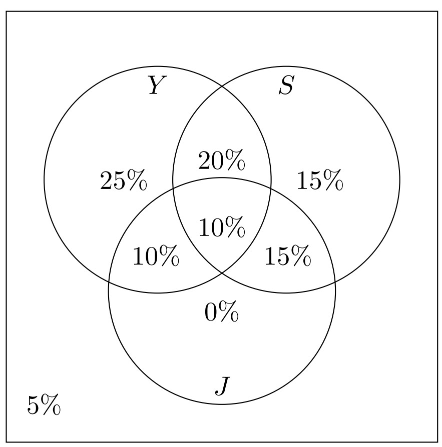
2. In a survey of student skills in a coding bootcamp, it was found that:

- 65% of students are proficient in Python
- 60% of students are proficient in SQL
- 35% of students are proficient in JavaScript
- 10% of students are proficient in all three languages
- 30% of students are proficient in both Python and SQL
- 25% of students are proficient in both SQL and JavaScript
- 20% of students are proficient in both Python and JavaScript

A student is selected at random. What is the probability they are proficient in exactly one of these three languages?

Solution:

Let Y, S, and J denote the events that a student is proficient in Python, SQL, and JavaScript respectively. We can draw the Venn diagram:



By the diagram, the required probability is

$$P(Y \text{ only}) + P(S \text{ only}) + P(J \text{ only}) = 25\% + 15\% + 0\% = \boxed{40\%}.$$

3. Consider the following experiment.

   1. Pick a random integer from 1 to 10 (inclusive).
   2. Toss a fair coin and record the outcome.

   Let A be the event the integer chosen is prime. Let B be the event the coin lands on tails.

   Are A and B independent? Mutually exclusive? Justify your answers.

Solution:

The sample space of the experiment is

$$\{(a,b) \mid a \in [1,10] \cap \mathbb{Z}, b \in \{H,T\}\}$$

Then, $A = \{(p,H),(p,T) \mid p \in \{2,3,5,7\}\}$ and $B = \{(a,T) \mid a \in [1,10] \cap \mathbb{Z}\}$. We have $P(A) = \frac{8}{20} = \frac{2}{5}$, $P(B) = \frac{10}{20} = \frac{1}{2}$, and $P(A \cap B) = \frac{4}{20} = \frac{1}{5}$. Since $P(A \cap B) = P(A)P(B)$, A and B are $\boxed{\text{independent}}$.

Also, since, for example, $(2,T) \in A$ and $(2,T) \in B$, i.e., $A \cap B \neq \emptyset$, A and B are $\boxed{\text{not mutually exclusive}}$.

4. A fair six-sided die is rolled 5 times, and the number faced up is recorded for each roll. What is the probability that the first and last rolls are even numbers, given that exactly three even numbers were rolled in total?

Solution:

Denote the events: F : "first roll is even", L : "last roll is even", and T : "exactly three rolls are even".

$$P(T) = \binom{5}{3} \cdot \left(\frac{1}{2}\right)^5 = \frac{5}{16},$$

$$P(F) = P(L) = \frac{1}{2}.$$

By independence, $P(F \cap L) = P(F)P(L) = \dfrac{1}{4}$.

Then, $P(F \cap L \mid T) = \dfrac{P(F \cap L \cap T)}{P(T)} = \dfrac{16}{5} P(F \cap L \cap T)$.

Note that $P(F \cap L \cap T) = \left(\dfrac{1}{2}\right) \cdot \dbinom{3}{1} \cdot \left(\dfrac{1}{2}\right)^3 \cdot \left(\dfrac{1}{2}\right) = \dfrac{3}{32}$.

Therefore, $P(F \cap L \mid T) = \dfrac{16}{5} \cdot \dfrac{3}{32} = \boxed{\dfrac{3}{10}}$.

5. A city's taxis are Red, Green, or Blue. The fleet consists of 84% Red taxis, 15% Green taxis, and 1% Blue taxis. A witness reports a taxi in a hit-and-run as being Green. Tests show that under similar conditions:

- If the taxi is Red, the witness says "Green" 10% of the time.
- If the taxi is Green, the witness says "Green" 80% of the time.
- If the taxi is Blue, the witness says "Green" 15% of the time.

What is the probability that the taxi was actually Red, given the witness identified it as Green?

Solution:

Let I be the event that the witness identifies the taxi as Green. Let R, G, and B be the events that the taxi is Red, Green, and Blue respectively.

We have:

$$P(R) = 0.84, \quad P(G) = 0.15, \quad P(B) = 0.01,$$
$$P(I|R) = 0.10, \quad P(I|G) = 0.80, \quad P(I|B) = 0.15.$$

By law of total probability:

$$\begin{aligned} P(I) &= P(R)P(I|R) + P(G)P(I|G) + P(B)P(I|B) \\ &= 0.84 \times 0.10 + 0.15 \times 0.80 + 0.01 \times 0.15 \\ &= 0.084 + 0.12 + 0.0015 \\ &= 0.2055. \end{aligned}$$

By Bayes' theorem:

$$\begin{aligned} P(R|I) &= \frac{P(R)P(I|R)}{P(I)} \\ &= \frac{0.84 \times 0.10}{0.2055} \\ &= \boxed{\frac{56}{137}}. \end{aligned}$$

6. Let X be a discrete random variable with the probability distribution $p(x)$ given below.

x	-2	0	1	3	4
$p(x)$	0.1	0.3	0.2	0.25	0.15

Compute $P(X \text{ is even})$ and $\mathbb{E}(X^2)$.

Solution:

We verify: $\sum_x p(x) = 0.1+0.3+0.2+0.25+0.15 = 1$ and $p(x) \in [0,1] \forall x$, so $p(x)$ is a valid probability distribution.

$$\begin{aligned} P(X \text{ is even}) &= p(-2) + p(0) + p(4) \\ &= 0.1 + 0.3 + 0.15 \\ &= \boxed{0.55}. \end{aligned}$$

$$\begin{aligned} \mathbb{E}(X^2) &= \sum_x x^2 p(x) \\ &= (-2)^2 \cdot 0.1 + 0^2 \cdot 0.3 + 1^2 \cdot 0.2 + 3^2 \cdot 0.25 + 4^2 \cdot 0.15 \\ &= 4 \cdot 0.1 + 0 + 1 \cdot 0.2 + 9 \cdot 0.25 + 16 \cdot 0.15 \\ &= 0.4 + 0 + 0.2 + 2.25 + 2.4 \\ &= \boxed{5.25}. \end{aligned}$$

7. A random variable X has the probability density function:

$$f(x) = \begin{cases} \frac{1}{36}x(6-x), & \text{for } 0 < x < 6, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Compute $P(1 \leq X \leq 2)$.

Solution:
We verify:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^6 \frac{1}{36}x(6-x)\,dx = \frac{1}{36}\left[3x^2 - \frac{1}{3}x^3\right]_0^6 = \frac{1}{36}(108-72) = 1.$$

and $f(x) \geq 0$ for all x. Thus, $f(x)$ is a valid probability density function.
Then,

$$P(1 \leq X \leq 2) = \int_1^2 f(x)\,dx = \int_1^2 \frac{1}{36}x(6-x)\,dx = \frac{1}{36}\left[3x^2 - \frac{1}{3}x^3\right]_1^2 = \boxed{\frac{5}{27}}.$$

(b) Compute $\text{Var}(3X+2)$.

Solution:

$$\mathbb{E}(X) = \int_0^6 xf(x)\,dx = \int_0^6 x \cdot \frac{1}{36}x(6-x)\,dx = \frac{1}{36}\left[2x^3 - \frac{1}{4}x^4\right]_0^6 = \frac{1}{36}(432-324) = 3.$$

$$\mathbb{E}(X^2) = \int_0^6 x^2 f(x)\,dx = \int_0^6 x^2 \cdot \frac{1}{36}x(6-x)\,dx = \frac{1}{36}\left[\frac{3}{2}x^4 - \frac{1}{5}x^5\right]_0^6 = \frac{54}{5}.$$

$$\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{54}{5} - 9 = \frac{9}{5}.$$

Thus,

$$\text{Var}(3X+2) = 9\text{Var}(X) = 9 \cdot \frac{9}{5} = \boxed{\frac{81}{5}}.$$

8. Roll a faire four-sided die twice. Let X be the number faced up on the first roll. Let Y be the maximum of the two rolls.

(a) Complete the joint distribution table for $x, y \in \{1, 2, 3, 4\}$.

Solution:
We draw the table of outcomes first.

2nd \ 1st	1	2	3	4
1	(1,1)	(1,2)	(1,3)	(1,4)
2	(2,1)	(2,2)	(2,3)	(2,4)
3	(3,1)	(3,2)	(3,3)	(3,4)
4	(4,1)	(4,2)	(4,3)	(4,4)

Then, counting the occurrences, we obtain the joint distribution table as follows.

Y \ X	1	2	3	4
1	$1/16$	0	0	0
2	$1/16$	$2/16$	0	0
3	$1/16$	$1/16$	$3/16$	0
4	$1/16$	$1/16$	$1/16$	$4/16$

(b) Compute $\mathbb{E}(XY)$.

Solution:

$$\begin{aligned}\mathbb{E}(XY) &= \sum_x \sum_y xy \cdot p(x, y) \\ &= \frac{1}{16} \cdot (1 + 2 + 3 + 4 + 8 + 6 + 8 + 27 + 12 + 64) \\ &= \boxed{\frac{135}{16}}.\end{aligned}$$