MATH1853 Linear Algebra, Probability & Statistics
25/26 Semester 1 – Part II
Assignment 1 (Proposed Solutions)

Remarks: *These are the proposed solutions of the author. They shall by no means be considered as an official solution provided by the department. They may contain errors. You are advised to use with discretion.*

1. For $z \neq 0$ and $\text{Arg}(z) \in (-\pi, \pi]$, the principal value of $\log z$ is defined by

$$\log(z) = \ln|z| + i\,\text{Arg}(z).$$

Compute $\log(z)$ for $z = \dfrac{1+i}{1-i}$.

Solution:
Simplify z:

$$z = \frac{1+i}{1-i} \cdot \frac{1+i}{1+i} = \frac{(1+i)^2}{1-i^2} = \frac{1+2i+i^2}{1+1} = \frac{2i}{2} = i.$$

Then, $|z| = \sqrt{(0)^2 + (1)^2} = 1$, and $\text{Arg}(z) = \dfrac{\pi}{2}$ since z lies on the positive imaginary axis.

By definition, $\log(z) = \ln 1 + i \cdot \dfrac{\pi}{2} = 0 + i \cdot \dfrac{\pi}{2} = \boxed{\dfrac{\pi}{2} i}$.

2. Use complex exponentials to prove the trigonometric identity.

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

Proof:
Using Euler's formula, we show that:

$$\begin{aligned}
\text{L.H.S.} &= \left(\frac{e^{iA} - e^{-iA}}{2i}\right)\left(\frac{e^{iB} - e^{-iB}}{2i}\right) \\
&= \frac{1}{4i^2}\left(e^{i(A+B)} - e^{i(A-B)} - e^{i(-A+B)} + e^{-i(A+B)}\right) \\
&= -\frac{1}{4}\left(e^{i(A+B)} + e^{-i(A+B)} - e^{i(A-B)} - e^{-i(A-B)}\right) \\
&= -\frac{1}{2}\left(\frac{e^{i(A+B)} + e^{-i(A+B)}}{2} - \frac{e^{i(A-B)} + e^{-i(A-B)}}{2}\right) \\
&= \frac{1}{2}[\cos(A-B) - \cos(A+B)] \\
&= \text{R.H.S.}
\end{aligned}$$

Q.E.D.

3. Let $z = x + yi$. Compute $\text{Im}(e^{z^2})$ as an elementary function of x and y.

Solution:
Evaluate z^2 first:

$$z^2 = (x+yi)^2 = x^2 + 2xyi + (yi)^2 = (x^2 - y^2) + 2xyi$$

Then:

$$\begin{aligned} e^{z^2} &= e^{(x^2-y^2)+2xyi} \\ &= e^{x^2-y^2} \cdot e^{2xyi} \\ &= e^{x^2-y^2} (\cos(2xy) + i\sin(2xy)) \\ &= e^{x^2-y^2}\cos(2xy) + ie^{x^2-y^2}\sin(2xy) \end{aligned}$$

Thus, the imaginary part is:

$$\text{Im}(e^{z^2}) = \boxed{e^{x^2-y^2}\sin(2xy)}$$

4. Let D be the set of complex numbers satisfying

$$\text{Im}(z) \le \text{Re}(z) \quad \text{and} \quad |\text{Re}(z)| \le 1.$$

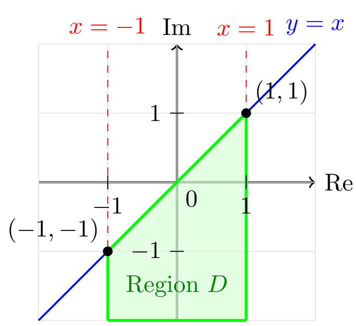
(a) For $z = x + yi$, define D by a Cartesian equation in set notation.

Solution:
$\text{Re}(z) = x$ and $\text{Im}(z) = y$, $x, y \in \mathbb{R}$. Thus,

$$D = \boxed{\{(x,y) \mid y \le x, \quad |x| \le 1, \quad x, y \in \mathbb{R}\}}$$

(b) Sketch the set D in the complex plane.

Solution:
By $y \le x$, the region is the area below the line $y = x$ with boundary included. By $|x| \le 1$, the region is bounded by the vertical lines $x = 1$ and $x = -1$ with boundaries included. Intersecting these two conditions, we have the shaded region below:



Set $D = \{(x,y) \mid -1 \le x \le 1, y \le x\}$

5. Find all solutions to $z^3 = (2+2i)^6$. Write your answer in set notation.

Solution:
For $w = 2+2i$, $|w| = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}$ and $\arg(w) = \tan^{-1}(1) = \frac{\pi}{4}$. Thus, $w = 2\sqrt{2}e^{i\pi/4}$, and $w^6 = (2\sqrt{2})^6 e^{i6\pi/4} = 512e^{i3\pi/2}$.
For $z^3 = re^{i\theta}$, we have

$$z_k = r^{1/3} e^{i(\theta+2k\pi)/3} \quad \text{for } k = 0, 1, 2.$$

Therefore, the solution set is given by

$$\boxed{\left\{8e^{i\pi/2}, 8e^{i7\pi/6}, 8e^{i11\pi/6}\right\}}$$

6. Let ω be a primitive n-th root of unity where $n > 1$. Evaluate the product

$$P = \omega \cdot \omega^2 \cdot \omega^3 \cdots \omega^{n-1}.$$

Express your answer in simplest form.

Solution:
Since ω is a primitive n-th root of unity, we can write

$$\omega = e^{\frac{2\pi k}{n} i} \qquad \text{where } k \in [1, n) \cap \mathbb{Z}$$

Thus,

$$\begin{aligned} P &= e^{\frac{2\pi k}{n} i} \cdot e^{\frac{4\pi k}{n} i} \cdot e^{\frac{6\pi k}{n} i} \cdots e^{\frac{2(n-1)\pi k}{n} i} \\ &= e^{\frac{2\pi k}{n} i \cdot (1+2+3+\cdots+(n-1))} \\ &= e^{\frac{2\pi k}{n} i \cdot \frac{(n-1)n}{2}} \\ &= e^{\pi k(n-1)i} \\ &= (-1)^{k(n-1)}. \end{aligned}$$

Observe that if n is even, then k must be odd (since $\gcd(k, n) = 1$). Then, $n-1$ is odd, and thus $k(n-1)$ is odd. Therefore, $P = -1$ when n is even. If n is odd, then $n-1$ is even, and thus $k(n-1)$ is even regardless of the parity of k. Therefore, $P = 1$ when n is odd. In conclusion, whether P is 1 or -1 depends on the parity of n.

$$P = \boxed{(-1)^{n-1}}.$$

7. Solve the complex equations. Write your answer in set notation.

(a) $z^2 + (1-2i)z - (1+i) = 0$

Solution:
We use the quadratic formula to solve for z:

$$\begin{aligned} z &= \frac{-(1-2i) \pm \sqrt{(1-2i)^2 - 4\cdot 1 \cdot (-(1+i))}}{2\cdot 1} \\ &= \frac{-1+2i \pm \sqrt{1-4i-4+4+4i}}{2} \\ &= \frac{-1+2i\pm 1}{2} \end{aligned}$$

Therefore, the solution set is

$$\boxed{\{i, -1+i\}}$$

(b) $z^2 = \overline{z}^2$

Solution:
Let $z = x + yi$ where $x, y \in \mathbb{R}$. Then, we have

$$\begin{aligned} (x+yi)^2 &= (x-yi)^2 \\ x^2 + 2xyi - y^2 &= x^2 - 2xyi + y^2 \\ xyi &= -xyi \\ 2xyi &= 0 \\ xyi &= 0 \\ xy &= 0 \end{aligned}$$

Therefore, we have the solution set:

$$\boxed{\{x + yi \mid xy = 0\}}$$

8. Let $D = \{z \in \mathbb{C} \mid \mathrm{Re}(z) = 4\}$ and define

$$f : D \to \mathbb{C} \quad \text{by} \quad f(z) = \sqrt{z}, \quad \text{the principal square root of } z.$$

Find the image set $f(D)$ defined by a well-known conic section.

Solution:
Let $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ where $r \geq 0$ and $\theta \in (-\pi, \pi]$.
$\mathrm{Re}(z) = 4 \implies r\cos\theta = 4.$
$f(z) = z^{\frac{1}{2}} = r^{\frac{1}{2}} e^{i\frac{\theta}{2}}.$
Note that since $\mathrm{Re}(z) = 4 > 0$, z must lie in the quadrants I or IV, so $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and $\arg(f(z)) \in (-\frac{\pi}{4}, \frac{\pi}{4})$.
Let $z = 4 + yi$, $y \in \mathbb{R}$, and $f(z) = a + bi$, $a, b \in \mathbb{R}$.

Then, $\sqrt{4+yi} = a + bi \Rightarrow 4 + yi = a^2 - b^2 + 2abi \Rightarrow \begin{cases} a^2 - b^2 = 4 \\ 2ab = y \end{cases}$.

Also, $a > 0$ since $\arg(f(z)) \in (-\frac{\pi}{4}, \frac{\pi}{4})$.
Therefore,

$$\boxed{f(D) = a + bi \mid a^2 - b^2 = 4, a > 0, a, b \in \mathbb{R}}.$$

The image set is $\boxed{\text{the right branch of the hyperbola } x^2 - y^2 = 4}$.