MATH1853 Linear Algebra, Probability & Statistics
25/26 Semester 1 – Part I
Assignment 3 (Proposed Solutions)

Remarks: *These are the proposed solutions of the author. They shall by no means be considered as an official solution provided by the department. They may contain errors. You are advised to use with discretion.*

1. Given the matrix $A = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$.

   (a) Find itw two eigenpairs with the eigenvectors normalized to unit vectors;

Solution:

$$\begin{aligned} \begin{vmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{vmatrix} &= 0 \\ (3-\lambda)(-\lambda) - (-2) &= 0 \\ \lambda^2 - 3\lambda + 2 &= 0 \\ (\lambda-1)(\lambda-2) &= 0 \\ \lambda_1 = 1, \lambda_2 &= 2 \end{aligned}$$

For $\lambda_1 = 1$,

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ -1 & -1 & 0 \end{array}\right] \xrightarrow[R_2 + 1/2 R_1]{1/2 R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Then, x_2 is free, and the eigenvector is

$$\mathbf{v_1} = x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}, x_2 \neq 0.$$

For convenience, we pick $x_2 = 1$, and normalize the eigenvector:

$$\widehat{\mathbf{v}}_\mathbf{1} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = 2$,

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ -1 & -2 & 0 \end{array}\right] \xrightarrow{R_2 + R_1} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Then, x_2 is free, and the eigenvector is

$$\mathbf{v_2} = x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}, x_2 \neq 0.$$

For convenience, we pick $x_2 = 1$, and normalize the eigenvector:

$$\widehat{\mathbf{v}}_\mathbf{2} = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Therefore, the two eigenpairs are

$$\boxed{(\lambda_1, \widehat{\mathbf{v}}_\mathbf{1}) = \left(1, \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right), \quad (\lambda_2, \widehat{\mathbf{v}}_\mathbf{2}) = \left(2, \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix}\right).}$$

   (b) Hence, or otherwise, compute A^{10};

Solution:

For $D = \begin{pmatrix} \mathbf{v_1} & \mathbf{v_2} \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$, and $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, by diagonalisation, we have

$$A = VDV^{-1} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}^{-1}.$$

Then,

$$A^{10} = VD^{10}V^{-1} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1^{10} & 0 \\ 0 & 2^{10} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

Calculating, we get

$$A^{10} = \boxed{\begin{pmatrix} 2047 & 2046 \\ -1023 & -1022 \end{pmatrix}}.$$

(c) Express the dominant direction of $A^{100}\mathbf{v}$ in a unit vector (where $\mathbf{v}$ is a random vector).

Solution:
Since $|\lambda_2| > |\lambda_1|$, the dominant direction of $A^{100}\mathbf{v}$ is along the eigenvector $\mathbf{v_2}$. Therefore, the normalised dominant direction is

$$\boxed{\begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}}$$

2. The vector $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is an eigenvector of the matrix $A = \begin{pmatrix} 1 & -1 & k \\ 1 & 3 & 0 \\ 1 & 1 & 2 \end{pmatrix}$.

(a) Find the value of the constant k and the corresponding eigenvalue λ.

Solution:
Since $\mathbf{v}$ is an eigenvector, we have $A\mathbf{v} = \lambda\mathbf{v}$.

$$A\mathbf{v} = \begin{pmatrix} 1 & -1 & k \\ 1 & 3 & 0 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2+k \\ -2 \\ 2 \end{pmatrix}$$

For this to equal $\lambda\mathbf{v} = \begin{pmatrix} \lambda \\ -\lambda \\ \lambda \end{pmatrix}$, we need:

$$2 + k = \lambda \tag{1}$$
$$-2 = -\lambda \tag{2}$$
$$2 = \lambda \tag{3}$$

From the second and third equations, $\lambda = 2$. Substituting into the first equation: $2 + k = 2$, so $\boxed{k = 0}$.

(b) What are all the eigenvalues of A?

Solution:
With $k = 0$, we have $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 3 & 0 \\ 1 & 1 & 2 \end{pmatrix}$. The characteristic polynomial is:

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} 1-\lambda & -1 & 0 \\ 1 & 3-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{pmatrix} \\ &= (2-\lambda)\det \begin{pmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{pmatrix} \\ &= (2-\lambda)[(1-\lambda)(3-\lambda) + 1] \\ &= (2-\lambda)(\lambda^2 - 4\lambda + 4) \\ &= (2-\lambda)(\lambda - 2)^2 = 0 \end{aligned}$$

The eigenvalues are $\boxed{\lambda = 2 \text{ (with algebraic multiplicity 3)}}$.

(c) Is A diagonalisable?

Solution:
Since $\lambda = 2$ is the only eigenvalue with algebraic multiplicity 3, A is diagonalisable if and only if the geometric multiplicity of $\lambda = 2$ is also 3, i.e., the dimension of the eigenspace is 3.

$$A - 2I = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The rank is 1, so dimension $= 3 - 1 = 2 < 3$. Therefore, $\boxed{A \text{ is not diagonalisable}}$.

3. Let $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$.

(a) What are the eigenvalues of A? Arrange them in ascending order.

Solution:
The characteristic polynomial is:

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{pmatrix} \\ &= (1-\lambda) \det \begin{pmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{pmatrix} - (-1) \det \begin{pmatrix} -1 & -1 \\ 0 & 1-\lambda \end{pmatrix} \\ &= (1-\lambda)[(2-\lambda)(1-\lambda) - 1] + [-(1-\lambda)] \\ &= (1-\lambda)[2 - 3\lambda + \lambda^2 - 1] - (1-\lambda) \\ &= (1-\lambda)[\lambda^2 - 3\lambda + 1 - 1] \\ &= (1-\lambda)\lambda(\lambda - 3) = 0 \end{aligned}$$

Therefore, the eigenvalues in ascending order are: $\boxed{0, 1, 3}$.

(b) Find the orthonormal matrix U in the eigendecomposition of A, i.e. $A = UDU^T$, where the eigenvalues in the matrix D are arranged in ascending order along the diagonal.

Solution:
For $\lambda = 0$: $(A - 0I)\mathbf{v} = \mathbf{0}$ gives $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, normalized: $\mathbf{u}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

For $\lambda = 1$: $(A - I)\mathbf{v} = \mathbf{0}$ gives $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, normalized: $\mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

For $\lambda = 3$: $(A - 3I)\mathbf{v} = \mathbf{0}$ gives $\mathbf{v}_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, normalized: $\mathbf{u}_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

Note that A is symmetric, and $\lambda_1 \neq \lambda_2 \neq \lambda_3$, so the set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ must be orthonormal.

Therefore, $\boxed{U = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}}$.

(c) If $B^3 = A$, what are the eigenvalues of B?

Solution:
If $B^3 = A$ and A has eigenvalues λ_A, then B has eigenvalues λ_B where $\lambda_B^3 = \lambda_A$.
For $\lambda_A = 0$: $\lambda_B = 0$.
For $\lambda_A = 1$: $\lambda_B = 1$.
For $\lambda_A = 3$: $\lambda_B = \sqrt[3]{3}$.
Therefore, the eigenvalues of B are: $\boxed{0, 1, \sqrt[3]{3}}$.

4. Let A be a 4×4 real symmetric matrix. Suppose $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 1 of A.

And A has another eigenvalue 2 whose geometri multiplicity is 3.

(a) Find an orthonormal basis for the eigenspace of eigenvalue 2. *(Hint: Eigenspaces of distinct eigenvalues of a symmetric matrix are orthogonal.)*

Solution:
Let $\mathbf{v}_2 = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^{\mathrm{T}}$ be an eigenvector of A corresponding to eigenvalue 2. Then we have

$$\mathbf{v}_1^{\mathrm{T}} \mathbf{v}_2 = \mathbf{v}_2^{\mathrm{T}} \mathbf{v}_1 = 0 \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \quad \Leftrightarrow \quad x_1 + x_4 = 0.$$

Any vector satisfying $x_1 + x_4 = 0$ is an eigenvector corresponding to eigenvalue 2. Thus, we take:

$$\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

Consider $\begin{bmatrix} \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \mid \mathbf{0} \end{bmatrix}$, we can easily verify that the three vectors are linearly independent.
Now we apply Gram-Schmidt process to orthonormalise the basis. Let $\hat{\mathbf{u}}_1 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}$ be a unit vector of the eigenspace ($\lambda = 2$), we get $\hat{\mathbf{u}}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^{\mathrm{T}}$.

Then, $\mathbf{u}_2 = \mathbf{v}_3 - \frac{\mathbf{v}_3^{\mathrm{T}} \hat{\mathbf{u}}_1}{\|\hat{\mathbf{u}}_1\|} \hat{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} - \mathbf{0} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \Rightarrow \hat{\mathbf{u}}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$.

Similarly, $\mathbf{u}_3 = \mathbf{v}_4 - \frac{\mathbf{v}_4^{\mathrm{T}} \hat{\mathbf{u}}_1}{\|\hat{\mathbf{u}}_1\|} \hat{\mathbf{u}}_1 - \frac{\mathbf{v}_4^{\mathrm{T}} \hat{\mathbf{u}}_2}{\|\hat{\mathbf{u}}_2\|} \hat{\mathbf{u}}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} - \mathbf{0} - \mathbf{0} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \hat{\mathbf{u}}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$.

Therefore, the orthonormal basis for the eigenspace of eigenvalue 2 is

$$\boxed{\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}}.$$

(b) Find the matrix A.

Solution:

Normalise $\mathbf{v}_1$ to get the unit eigenvector $\hat{\mathbf{u}}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Construct the orthonormal matrix $U = \begin{bmatrix} \hat{\mathbf{u}}_1 & \hat{\mathbf{u}}_2 & \hat{\mathbf{u}}_3 & \hat{\mathbf{u}}_4 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$.

Also construct $D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

Then, $A = UDU^{\mathrm{T}} = \left(\frac{1}{\sqrt{2}}\right)^2 \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{3}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{3}{2} \end{bmatrix}}$