MATH1853 Linear Algebra, Probability & Statistics
25/26 Semester 1 – Part I
Assignment 2 (Proposed Solutions)

Remarks: *These are the proposed solutions of the author. They shall by no means be considered as an official solution provided by the department. They may contain errors. You are advised to use with discretion.*

1. Calculate the determinant of matrix A:

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 2 & -1 \\ -1 & 0 & 0 & 2 \end{bmatrix}$$

Solution:

By expanding along the first column, we have:

$$\begin{aligned} \det(A) &= (+1) \cdot 2 \cdot \begin{vmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{vmatrix} + (-1) \cdot (-1) \cdot \begin{vmatrix} -1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 2 & -1 \end{vmatrix} \\ &= 2 \cdot (2 \cdot 2 \cdot 2) + 1 \cdot (-1) \cdot (-1) \cdot (-1) \\ &= 16 - 1 \\ &= \boxed{15} \end{aligned}$$

2. Calculate the determinant of the following $n \times n$ matrix:

$$D_n = \begin{bmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a \end{bmatrix}$$

Solution:

By performing row operations and update the determinant accordingly, we have

$$\begin{aligned}
\det(D_n) &= \det \begin{bmatrix} a & b & b & \cdots & b \\ b-a & a-b & 0 & \cdots & 0 \\ b-a & 0 & a-b & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b-a & 0 & 0 & \cdots & a-b \end{bmatrix} \quad (R_n \leftarrow R_n - R_1, n \in [2, n]) \\
&= (a-b)^{n-1} \cdot \det \begin{bmatrix} a & b & b & \cdots & b \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{bmatrix}^{\mathrm{T}} \quad (\text{factorise } (a-b) \text{ from } R_n, n \in [2, n]) \\
&= (a-b)^{n-1} \cdot \det \begin{bmatrix} a & -1 & -1 & \cdots & -1 \\ b & 1 & 0 & \cdots & 0 \\ b & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & 0 & 0 & \cdots & 1 \end{bmatrix} \quad (\det(A) = \det(A^{\mathrm{T}})) \\
&= (a-b)^{n-1} \cdot \det \begin{bmatrix} a+(n-1)b & 0 & 0 & \cdots & 0 \\ b & 1 & 0 & \cdots & 0 \\ b & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & 0 & 0 & \cdots & 1 \end{bmatrix} \quad (R_1 \leftarrow R_1 + \sum_{n=2}^{n} R_n) \\
&= (a-b)^{n-1} \cdot (a+(n-1)b) \cdot \det(I_{n-1}) \quad (\text{expand along } R_1) \\
&= (a-b)^{n-1} \cdot (a+(n-1)b) \cdot 1 \\
&= \boxed{(a-b)^{n-1}(a+(n-1)b)}
\end{aligned}$$

3. Given four vectors $\alpha_1 = \begin{bmatrix}1\\4\\0\\2\end{bmatrix}, \alpha_2 = \begin{bmatrix}2\\7\\1\\3\end{bmatrix}, \alpha_3 = \begin{bmatrix}0\\1\\-1\\a\end{bmatrix}, \beta = \begin{bmatrix}3\\10\\b\\4\end{bmatrix}$, please find:

(a) for what values of a and b is the β column linearly independent from $\alpha_1, \alpha_2, \alpha_3$;

Solution:

In this solution, row operation notations will always place the resulting row on the left-hand side. For example, $R_1 - R_2$ means the new R_1 is obtained by subtracting R_2 from the old R_1. Swapping is denoted by $R_i \leftrightarrow R_j$.

When the system $[\alpha_1\,\alpha_2\,\alpha_3\,|\,\beta]$ does not have a non-trivial solution, β is linearly independent from $\alpha_1, \alpha_2, \alpha_3$.

$$\left[\begin{array}{ccc|c}1 & 2 & 0 & 3\\4 & 7 & 1 & 10\\0 & 1 & -1 & b\\2 & 3 & a & 4\end{array}\right] \xrightarrow[R_4-2R_1]{R_2-4R_1} \left[\begin{array}{ccc|c}1 & 2 & 0 & 3\\0 & -1 & 1 & -2\\0 & 1 & -1 & b\\0 & -1 & a & -2\end{array}\right] \xrightarrow[R_4-R_2]{-R_2, R_3+R_2} \left[\begin{array}{ccc|c}1 & 2 & 0 & 3\\0 & 1 & -1 & 2\\0 & 0 & 0 & b-2\\0 & 0 & a-1 & 0\end{array}\right]$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \left[\begin{array}{ccc|c}1 & 2 & 0 & 3\\0 & 1 & -1 & 2\\0 & 0 & a-1 & 0\\0 & 0 & 0 & b-2\end{array}\right]$$

The system does not have a non-trivial solution when $b - 2 \neq 0 \Rightarrow b \neq 2$.

Therefore, $\boxed{\forall a \in \mathbb{R}, b \neq 2}$, we always have $\beta \notin \text{span}\{\alpha_1, \alpha_2, \alpha_3\}$.

(b) for what values of a and b can β be uniquely linearly represented by $\alpha_1, \alpha_2, \alpha_3$, i.e., a unique $\mathbf{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ such that $\beta = \begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{bmatrix}\mathbf{x}$. Please write the corresponding expression.

Solution:

Consider the linear system from the previous part. From R_4, we have $0x_3 = b - 2 \Rightarrow b = 2$. For the system to have a unique solution, we must have $a - 1 \neq 0 \Rightarrow a \neq 1$.

We pick $a = b = 2$ and solve the system:

$$\left[\begin{array}{ccc|c}1 & 2 & 0 & 3\\0 & 1 & -1 & 2\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{array}\right] \xrightarrow[R_1-2R_2]{R_2+R_3} \left[\begin{array}{ccc|c}1 & 0 & 2 & -1\\0 & 1 & 0 & 2\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{array}\right] \xrightarrow{R_1-2R_3} \left[\begin{array}{ccc|c}1 & 0 & 0 & -1\\0 & 1 & 0 & 2\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{array}\right]$$

Therefore, the unique solution is $\boxed{\mathbf{x} = \begin{bmatrix}-1\\2\\0\end{bmatrix}}$, and the required values are $\boxed{a \neq 1, b = 2}$.

4. Given five vectors:

$$\alpha_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 4 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 0 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \quad \alpha_3 = \begin{bmatrix} 3 \\ 0 \\ 7 \\ 14 \end{bmatrix}, \quad \alpha_4 = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}, \quad \alpha_5 = \begin{bmatrix} 2 \\ 1 \\ 5 \\ 10 \end{bmatrix}$$

Find one maximal linearly independent set of this vector group when we choose sequentially from α_1 to α_5, then put down the rank of this set.

Solution:

We form the augmented matrix with the five vectors as columns:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 3 & 1 & 2 & 0 \\ -1 & 3 & 0 & -2 & 1 & 0 \\ 2 & 1 & 7 & 2 & 5 & 0 \\ 4 & 2 & 14 & 0 & 10 & 0 \end{array}\right] \xrightarrow[R_4-4R_1]{R_3-2R_1, R_2+R_1} \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 1 & 2 & 0 \\ 0 & 3 & 3 & -1 & 3 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & -4 & 2 & 0 \end{array}\right] \xrightarrow[R_2-3R_3, R_4-2R_3]{R_2 \leftrightarrow R_3} \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[-1/4R_4]{-R_3} \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \xrightarrow[R_4-R_3]{R_1-R_3} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1-R_3} \left[\begin{array}{ccccc|c} \boxed{1} & 0 & 0 & 0 & -1 & 0 \\ 0 & \boxed{1} & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, the maximal linearly independent set is $\boxed{\{\alpha_1, \alpha_2, \alpha_4\}}$, and the rank of this set is $\boxed{3}$.

5. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation defined by

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} x_1 + 2x_2 + 3x_3 - x_4 \\ 3x_1 + 5x_2 + 8x_3 - 2x_4 \\ x_1 + x_2 + 2x_3 \end{bmatrix}$$

(a) Find a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

Solution:

By inspection, we have $\boxed{A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 3 & 5 & 8 & -2 \\ 1 & 1 & 2 & 0 \end{bmatrix}}$.

(b) Find a set of basis vectors spanning the nullspace of A. You may want to standardise your answer by using our in-class free-column(s) approach, i.e., one-hotting each free column respectively, to locate the nullspace vectors.

Solution:

We find $\mathbf{x_N} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ such that $T(\mathbf{x_N}) = A\mathbf{x_N} = 0$.

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & -1 & 0 \\ 3 & 5 & 8 & -2 & 0 \\ 1 & 1 & 2 & 0 & 0 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 3R_1} \left[\begin{array}{cccc|c} 1 & 2 & 3 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & -1 & 1 & 0 \end{array}\right] \xrightarrow[-R_2]{R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 2 & 3 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 - 2R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Solving the system, we have

$$\mathbf{x_N} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Then, we examine the linear independence of the set:

$$\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

by constructing another augmented matrix:

$$\left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{-R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{array}\right] \xrightarrow[R_4 + R_1 - R_2]{R_3 - R_1 - R_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore, the set is linearly independent and forms a basis of the nullspace of A. We conclude that a basis spanning the nullspace of A is

$$\boxed{\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}}.$$