MATH1853 Linear Algebra, Probability & Statistics
25/26 Semester 1 – Part I
Assignment 1 (Proposed Solutions)

Remarks: *These are the proposed solutions of the author. They shall by no means be considered as an official solution provided by the department. They may contain errors. You are advised to use with discretion.*

1. Solve the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & -2 & -2 & -2 \\ 0 & -2 & 0 & -1 \\ 1 & 2 & 2 & 0 \\ 1 & -2 & 1 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -31 \\ -19 \\ 27 \\ -4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

Solution:

In this solution, row operation notations will always place the resulting row on the left-hand side. For example, $R_1 - R_2$ means the new R_1 is obtained by subtracting R_2 from the old R_1. Swapping is denoted by $R_i \leftrightarrow R_j$.

Construct the augmented matrix $[A|\mathbf{b}]$ and reduce it to the row echelon form:

$$\left[\begin{array}{cccc|c} 1 & -2 & -2 & -2 & -31 \\ 0 & -2 & 0 & -1 & -19 \\ 1 & 2 & 2 & 0 & 27 \\ 1 & -2 & 1 & 2 & -4 \end{array}\right] \xrightarrow[R_4 - R_1]{\substack{R_1 - R_2, \\ R_3 - R_1}} \left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & -12 \\ 0 & -2 & 0 & -1 & -19 \\ 0 & 4 & 4 & 2 & 58 \\ 0 & 0 & 3 & 4 & 27 \end{array}\right] \xrightarrow[R_3 + 2R_2]{-1/2 R_2} \left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & -12 \\ 0 & 1 & 0 & 1/2 & 19/2 \\ 0 & 0 & 4 & 0 & 20 \\ 0 & 0 & 3 & 4 & 27 \end{array}\right]$$

$$\xrightarrow{1/4 R_3} \left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & -12 \\ 0 & 1 & 0 & 1/2 & 19/2 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 3 & 4 & 27 \end{array}\right] \xrightarrow{R_4 - 3R_3} \left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & -12 \\ 0 & 1 & 0 & 1/2 & 19/2 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 4 & 12 \end{array}\right] \xrightarrow[1/4 R_4]{R_1 + 2R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 1/2 & 19/2 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow[R_2 - 1/2 R_4]{R_1 + R_4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

Therefore, we have

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 8 \\ 5 \\ 3 \end{bmatrix}}$$

2. Solve the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 1 & -3 & 2 & 7 \\ -1 & -1 & 3 & 1 & -4 \\ 2 & 1 & -5 & 2 & 10 \\ 2 & 1 & -5 & 0 & 8 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ -2 \\ 6 \\ 4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}.$$

Solution:
Again, construct the augmented matrix and perform row operations:

$$\left[\begin{array}{ccccc|c} 1 & 1 & -3 & 2 & 7 & 5 \\ -1 & -1 & 3 & 1 & -4 & -2 \\ 2 & 1 & -5 & 2 & 10 & 6 \\ 2 & 1 & -5 & 0 & 8 & 4 \end{array}\right] \xrightarrow[R_3-2R_1,\; R_4-2R_1]{R_2+R_1} \left[\begin{array}{ccccc|c} 1 & 1 & -3 & 2 & 7 & 5 \\ 0 & 0 & 0 & 3 & 3 & 3 \\ 0 & -1 & 1 & -2 & -4 & -4 \\ 0 & -1 & 1 & -4 & -6 & -6 \end{array}\right]$$

$$\xrightarrow[1/3R_2\leftrightarrow R_4]{-R_3, R_1+R_3} \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & 3 & 1 \\ 0 & -1 & 1 & -4 & -6 & -6 \\ 0 & 1 & -1 & 2 & 4 & 4 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow[-R_2]{R_3+R_2} \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & 3 & 1 \\ 0 & 1 & -1 & 4 & 6 & 6 \\ 0 & 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_4+1/2R_3]{-1/2R_3} \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & 3 & 1 \\ 0 & 1 & -1 & 4 & 6 & 6 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2-4R_3} \left[\begin{array}{ccccc|c} \boxed{1} & 0 & -2 & 0 & 3 & 1 \\ 0 & \boxed{1} & -1 & 0 & 2 & 2 \\ 0 & 0 & 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

From the final matrix, we have the pivot columns 1, 2, and 4, and hence the free variables are x_3 and x_5. The solution $\mathbf{x}$ is in the form $\mathbf{x} = \mathbf{x_p} + x_3\mathbf{v_1} + x_5\mathbf{v_2}$.

For the particular solution $\mathbf{x_p}$, we set $x_3 = 0$ and $x_5 = 0$, and solve for the pivot variables:

$$\begin{cases} x_1 - 2(0) + 3(0) = 1 \\ x_2 - (0) + 2(0) = 2 \\ x_4 + (0) = 1 \end{cases} \Rightarrow \begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_4 = 1 \end{cases} \Rightarrow \mathbf{x_p} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

For $\mathbf{v_1}$ corresponding to x_3, we set $x_3 = 1$ and all other free variables to 0, and solve for the pivot variables:

$$\begin{cases} x_1 - 2(1) + 3(0) = 0 \\ x_2 - (1) + 2(0) = 0 \\ x_4 + (0) = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 2 \\ x_2 = 1 \\ x_4 = 0 \end{cases} \Rightarrow \mathbf{v_1} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_3\mathbf{v_1} = x_3 \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Similarly, for $\mathbf{v_2}$, we have:

$$\mathbf{v_2} = \begin{bmatrix} -3 \\ -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

Therefore, the solution is

$$\boxed{\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad x_3, x_5 \in \mathbb{R}}.$$

3. Suppose that the following matrix A is the augmented matrix for a system of linear equations, where α is a real number. Determine all values of α such that the system is consistent.

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & -1 & -2 & \alpha^2 \\ -1 & -7 & -11 & \alpha \end{array}\right]$$

Solution:

Perform row operations on A:

$$A \xrightarrow[R_3+R_1]{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -5 & -8 & \alpha^2-8 \\ 0 & -5 & -8 & \alpha+4 \end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -5 & -8 & \alpha^2-8 \\ 0 & 0 & 0 & -\alpha^2+\alpha+12 \end{array}\right]$$

Consider R_3. The system is inconsistent if and only if the last entry of R_3 is non-zero, i.e.

$$\begin{aligned} -\alpha^2+\alpha+12 &\neq 0 \\ -(\alpha-4)(\alpha+3) &\neq 0 \\ \alpha &\neq 4, -3 \end{aligned}$$

Then, the system is consistent if and only if $\boxed{\alpha = 4 \text{ or } \alpha = -3}$.

4. Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}, \mathbf{v_5}\}$ where

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{v_2} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v_3} = \begin{bmatrix} 1 \\ 5 \\ -1 \\ 5 \end{bmatrix}, \quad \mathbf{v_4} = \begin{bmatrix} 1 \\ 1 \\ 4 \\ -1 \end{bmatrix}, \quad \mathbf{v_5} = \begin{bmatrix} 2 \\ 7 \\ 0 \\ 2 \end{bmatrix}.$$

Find the basis vectors for Span(S) from the set S, choosing sequentially from $\mathbf{v_1}$ onwards.

Solution:

Construct the augmented matrix with the vectors as columns, and row reduce:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 2 & 0 \\ 2 & 3 & 5 & 1 & 7 & 0 \\ 2 & 1 & -1 & 4 & 0 & 0 \\ -1 & 1 & 5 & -1 & 2 & 0 \end{array}\right] \xrightarrow[R_4+R_1]{R_2-2R_1,\ R_3-2R_1} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 3 & -1 & 3 & 0 \\ 0 & -1 & -3 & 2 & -4 & 0 \\ 0 & 2 & 6 & 0 & 4 & 0 \end{array}\right]$$

$$\xrightarrow[R_4+2R_3]{R_3+R_2} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 3 & -1 & 3 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 4 & -4 & 0 \end{array}\right] \xrightarrow{R_4-4R_3} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 3 & -1 & 3 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_1-R_3]{R_2+R_3} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 3 & 0 \\ 0 & 1 & 3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1-R_2} \left[\begin{array}{ccccc|c} \boxed{1} & 0 & -2 & 0 & 1 & 0 \\ 0 & \boxed{1} & 3 & 0 & 2 & 0 \\ 0 & 0 & 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The pivot columns are the 1st, 2nd, and 4th columns, indicating that $\mathbf{v_1}$, $\mathbf{v_2}$, and $\mathbf{v_4}$ are linearly independent, and form a basis for Span(S).

Also note that $\mathbf{v_3} = -2\mathbf{v_1} + 3\mathbf{v_2}$ and $\mathbf{v_5} = \mathbf{v_1} + 2\mathbf{v_2} - \mathbf{v_4}$.

Therefore, a basis for Span(S) is $\boxed{\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_4}\}}$.

5. Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$, where

$$\mathbf{v_1} = \begin{bmatrix}1\\3\\1\end{bmatrix}, \quad \mathbf{v_2} = \begin{bmatrix}1\\a\\4\end{bmatrix}, \quad \mathbf{v_3} = \begin{bmatrix}0\\2\\b\end{bmatrix}.$$

Here, a and b are scalars. Find b in terms of a such that the vectors are linearly dependent.

Solution:

The vectors are linearly dependent if there exist scalars c_1, c_2, c_3, not all zero, such that

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3} = \mathbf{0} \qquad \text{——} \ (1)$$

We construct the augmented matrix and row reduce:

$$\left[\begin{array}{ccc|c}1 & 1 & 0 & 0\\3 & a & 2 & 0\\1 & 4 & b & 0\end{array}\right] \xrightarrow[R_3-R_1]{R_2-3R_1} \left[\begin{array}{ccc|c}1 & 1 & 0 & 0\\0 & a-3 & 2 & 0\\0 & 3 & b & 0\end{array}\right] \xrightarrow{1/3R_3} \left[\begin{array}{ccc|c}1 & 1 & 0 & 0\\0 & a-3 & 2 & 0\\0 & 1 & b/3 & 0\end{array}\right] \xrightarrow{R_2\leftrightarrow R_3} \left[\begin{array}{ccc|c}1 & 1 & 0 & 0\\0 & 1 & b/3 & 0\\0 & a-3 & 2 & 0\end{array}\right]$$

$$\xrightarrow{R_1-R_2} \underbrace{\left[\begin{array}{ccc|c}1 & 0 & -b/3 & 0\\0 & 1 & b/3 & 0\\0 & a-3 & 2 & 0\end{array}\right]}_{(*)} \xrightarrow[\text{assume } a\neq 3]{1/(a-3)R_3} \left[\begin{array}{ccc|c}1 & 0 & -b/3 & 0\\0 & 1 & b/3 & 0\\0 & 1 & 2/a-3 & 0\end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{ccc|c}1 & 0 & -b/3 & 0\\0 & 1 & b/3 & 0\\0 & 0 & 2/a-3 - b/3 & 0\end{array}\right]$$

For the vectors to be linearly dependent, we need the last row to be all zeros, i.e.

$$\begin{aligned}\frac{2}{a-3} - \frac{b}{3} &= 0\\ \frac{2}{a-3} &= \frac{b}{3}\\ b &= \frac{6}{a-3}.\end{aligned}$$

We also inspect the case when $a = 3$. (*) becomes

$$\left[\begin{array}{ccc|c}1 & 0 & -b/3 & 0\\0 & 1 & b/3 & 0\\0 & 0 & 2 & 0\end{array}\right]$$

which means that (1) has only the trivial solution $c_1 = c_2 = c_3 = 0$. Thus, the vectors are linearly independent when $a = 3$. Then, when the set is linearly dependent, we must have $a \neq 3$.

Therefore, the final answer is

$$b = \boxed{\frac{6}{a-3}, \quad a \neq 3.}$$