COMP2121 Discrete Mathematics

25/26 Semester 1

Assignment 3

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1. Basics Probabilities and Random Variables

(a) [4 points] There are 5 boxes, 3 of which contain 10 HKD each, and 2 of which are empty. Bob will randomly select 3 boxes, and he will win the total amount of money in the selected boxes. What is the probability that he will win at least 20 HKD?

Answers:

Let X be the total amount of money Bob wins. $P(X \geq 20) = 1 - P(X=0) - P(X=10) = 1 - 0 - \frac{\binom{3}{2}}{\binom{5}{3}} = 1 - \frac{3}{10} = \boxed{\frac{7}{10}}$.

(b) [4 points] Suppose you toss a fair coin for multiple times. Define L to be the number of tosses you have to make until either of the following is satisfied:

(1) The number of "head" has reached 3.

(2) The total number of tosses reaches 5.

Determine $\mathbb{E}(L)$.

Answers:

If $L = 3$: $P(L=3) = (0.5)^3 = 0.125$.

If $L = 4$: $P(L=4) = \binom{3}{2}(0.5)^3(0.5) = 0.1875$.

If $L = 5$: The 5th toss is irrelevant. $P(L=5) = (0.5)^4 + \binom{4}{1}(0.5)^3(0.5) + \binom{4}{2}(0.5)^2(0.5)^2 = 0.6875$

$$\mathbb{E}(L) = 3 \cdot P(L=3) + 4 \cdot P(L=4) + 5 \cdot P(L=5) = \boxed{4.5625}$$

(c) [4 points] The joint distribution $P(X, Y)$ of two random variables X and Y is shown in the following table:

X \ Y	0	1	2
-1	0	1/6	1/6
0	1/6	1/6	0
1	0	1/6	1/6

Show that X and Y are uncorrelated but not independent.

Answers:

Marginal probabilities:

$$P_X(x) = \sum_{\text{all } y} P(X=x, Y=y) = \begin{cases} \frac{2}{6}, & x = -1 \\ \frac{2}{6}, & x = 0 \\ \frac{2}{6}, & x = 1 \end{cases} = \frac{1}{3}$$

and

$$P_Y(y) = \sum_{\text{all } x} P(X=x, Y=y) = \begin{cases} \frac{1}{6}, & y = 0 \\ \frac{1}{2}, & y = 1 \\ \frac{1}{3}, & y = 2 \end{cases}.$$

Then,

$$\mathbb{E}(X) = \sum_{\text{all } x} x \cdot P_X(x) = 0 \qquad \mathbb{E}(Y) = \sum_{\text{all } y} y \cdot P_Y(y) = \frac{1}{2} + 2 \cdot \frac{1}{3} = \frac{7}{6} \qquad \mathbb{E}(XY) = \sum_{\text{all } x}\sum_{\text{all } y} xy \cdot P(x,y) = 0.$$

Also,

$$\text{Cov}(XY) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 0$$

Hence, $\boxed{X \text{ and } Y \text{ are not correlated}}$.

However, $P(-1, 0) = 0 \neq \frac{1}{18} = P_X(-1)P_Y(0)$. Therefore, $\boxed{X \text{ and } Y \text{ are not independent}}$.

(d) [4 points] Alice and Bob are playing a game. At the start, Alice randomly chooses a real number $x \in [0, 5]$. The game consists of n rounds. In each round, Alice secretly tosses a fair coin. If the outcome is "head", she tells Bob $X_i = x + 1$, otherwise, she tells him $X_i = x - 1$. At the end, Bob computes the estimate

$$\hat{x} = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

Determine how large n should be to ensure that $|\hat{x} - x| < \frac{1}{2}$ with a probability of at least 0.9.

Answers:
Let $Y_i = X_i - x$. Since $X_i = x \pm 1$ with probability 0.5, Y_i takes values ± 1 with probability 0.5.

$$\mathbb{E}(Y_i) = 1(0.5) + (-1)(0.5) = 0, \quad \mathrm{Var}(Y_i) = \mathbb{E}(Y_i^2) - (\mathbb{E}(Y_i))^2 = 1 - 0 = 1.$$

Let $\overline{Y} = \hat{x} - x = \frac{1}{n}\sum_{i=1}^{n}(X_i - x) = \frac{1}{n}\sum_{i=1}^{n} Y_i$. Then

$$\mathbb{E}(\overline{Y}) = 0, \quad \mathrm{Var}(\overline{Y}) = \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{Var}(Y_i) = \frac{n}{n^2} = \frac{1}{n}.$$

We want to find n such that

$$P\left(|\hat{x} - x| < \frac{1}{2}\right) \geq 0.9 \iff 1 - P\left(|\hat{x} - x| \geq \frac{1}{2}\right) \geq 0.9$$
$$\iff P\left(|\overline{Y}| \geq \frac{1}{2}\right) \leq 0.1.$$

Using Chebyshev's inequality,

$$P\left(|\overline{Y}| \geq \frac{1}{2}\right) \leq \frac{\mathrm{Var}(\overline{Y})}{(1/2)^2} = \frac{1/n}{1/4} = \frac{4}{n}.$$

To ensure the condition holds, we require

$$\frac{4}{n} \leq 0.1 \implies n \geq 40.$$

Thus, n should be at least $\boxed{40}$.

2. Basics on Graphs

(a) [6 points] Let $V = \{a, b, c, d\}$ and $E = \{(a, b), (a, c), (a, d), (d, c)\}$. Count the number of subgraphs of the graph $G = (V, E)$.

Answers:
By cases:

(1) Null graph: 1 subgraph.

(2) $V' = \{a, b, c, d\} \subseteq V$: $2^{|E|} = 2^4 = 16$ subgraphs.

(3) $V' = \{a, b, c\} \subset V$: $2^2 = 4$ subgraphs.

(4) $V' = \{a, b, d\} \subset V$: $2^2 = 4$ subgraphs.

(5) $V' = \{a, c, d\} \subset V$: $2^3 = 8$ subgraphs.

(6) $V' = \{b, c, d\} \subset V$: $2^1 = 2$ subgraphs.

(7) $V' = \{a, b\}, \{a, c\}, \{a, d\} \subset V$: $2 \cdot 3 = 6$ subgraphs.

(8) $V' = \{b, c\}, \{b, d\} \subset V$: $1 \cdot 2 = 2$ subgraphs.

(9) $V' = \{c, d\} \subset V$: $2^1 = 2$ subgraphs.

(10) $V' = \{a\}, \{b\}, \{c\}, \{d\} \subset V$: $1 \cdot 4 = 4$ subgraphs.

Therefore, there are $\boxed{49}$ subgraphs.

(b) [4 points] Draw a simple, connected, undirected graph consisting of 1 vertex of degree 1, 2 vertices of degree 2, 3 vertices of degree 3, and 4 vertices of degree 4. Is the graph you draw planar or not? Justify your answer.

Answers:

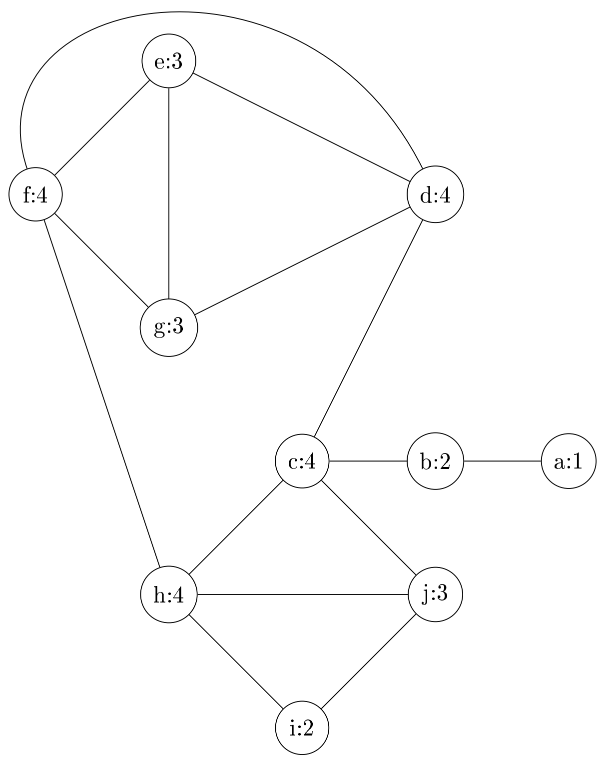

Figure 1: Graph for Question (2)(b)

The required graph is shown on the left. As visually presented, the graph satisfies all the requirements and has no crossing edges.

Also, we find that $v = |V| = 10$, $e = |E| = 15$, and $r = 7$. By Euler's formula, $v + r - e = 2$ holds. Therefore, the graph is $\boxed{\text{planar}}$.

(c) [6 points] How many simple, undirected graphs that do not contain any cycle can you construct with a fixed vertex set $\{A, B, C, D, E\}$? Explain your answer.

Answers:

The original solution proposed by the author was incorrect/incomplete (graded 4/6). It is removed in this document.

(d) [4 points] The diameter of a graph is defined to be the length of the shortest path between the most distanced vertices. What is the diameter of the graph you drew in Question (2)(b)?

Answers:

We start from vertex a (degree 1) and perform Breadth-First Search (BFS) to find the furthest vertex. Then, we have:

Distance	1	2	3	4
Vertices	b	c	d, g, h, j	e, f, g, i

The furthest vertices from a are e, f, g, i with distance 4. Also, to get to a from any vertex other than b, at least 2 edges are needed. Therefore, the diameter of the graph is $\boxed{4}$.

3. Proofs with Graphs

(a) [6 points] Let G be a simple, undirected graph with at least two vertices. If G has exactly one vertex of degree 1, show that there exists a cycle in G.

Proof:

Assume, for contradiction, that G is acyclic. Then, G is a tree. In a tree, there are at least two vertices of degree 1 (the leaves). This contradicts the assumption that G has exactly one vertex of degree 1. Therefore, our assumption is false, and G must contain at least one cycle.

Q.E.D.

(b) [8 points] A subset set of edges E' is called an edge cut of a graph $G = (V, E)$ if the subgraph $(V, E - E')$ of G is disconnected. The edge connectivity of a graph is the minimum number of edges in an edge cut. Suppose G is an undirected graph with connectivity 2 and at least 3 vertices. Prove that there exists a cycle in G.

Proof:

Assume, for contradiction, that G is acyclic. Then G is a forest.

Since G has at least 3 vertices and is connected (edge connectivity of 2 implies the graph is connected), G must be a tree.

In any tree, there exists at least one edge e such that removing e disconnects the tree into exactly two components. In fact, every edge in a tree is a bridge, meaning removing any single edge disconnects the graph.

This implies that the edge connectivity of G is 1 (we can disconnect G by removing just one edge), which contradicts the assumption that G has edge connectivity 2.

Therefore, our assumption is false, and G must contain at least one cycle.

Q.E.D.

4. Generalized Square Graphs

Consider the family of simple, undirected graphs $G_{m,n} = (V_{m,n}, E_{m,n})$, where m, n are positive integers. The set of vertices is defined by $V_{m,n} = \{(i, j) \in \mathbb{Z}^2 : 1 \le i \le m, 1 \le j \le n\}$, and the set of edges is defined by the following property: for any two distinct vertices $v_1 = (i_1, j_1), v_2 = (i_2, j_2) \in V_{m,n}$, $\{v_1, v_2\} \in E_{m,n}$ if and only if $i_1 = i_2$ or $j_1 = j_2$.

(a) [4 points] Determine the number of edges in $G_{m,n}$ for generic m, n.

Answers:

On the i-th row, there are vertices $(i, 1), (i, 2), \dots, (i, n)$, totalling n vertices. On the j-th column, there are vertices $(1, j), (2, j), \dots, (m, j)$, totalling m vertices. In any row, each vertex connects to $n - 1$ other vertices; in each column, each vertex connects to $m - 1$ other vertices. Therefore, $\deg(v) = m + n - 2$ for any vertex $v \in V_{m,n}$.

By Handshaking, $\sum \deg(v) = 2|E_{m,n}| \Rightarrow |E_{m,n}| = \frac{1}{2} \sum \deg(v) = \boxed{\frac{1}{2} mn(m + n - 2)}$.

(b) [6 points] Is $G_{3,3}$ planar? Prove your statement.

Proof:

Consider the following arrangement of $G_{3,3}$:

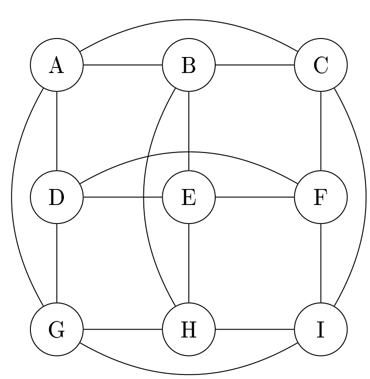

Figure 2: An arrangement of $G_{3,3}$

Choose two sets of vertices: $\{A, E, I\}$ and $\{B, D, F\}$. We can directly make connections: AB, AD, EB, ED, EF, IF.

Other connections can be made via intermediate vertices: $AI = AC \to CI, IB = IH \to HB, ID = IG \to GD$. Then, we have found a subdivision of $K_{3,3}$ in $G_{3,3}$. By Kuratowski's Theorem, $\boxed{G_{3,3} \text{ is non-planar}}$.

Q.E.D.

(c) [6 points] Find all (m, n) such that $G_{m,n}$ is planar.

Answers:
The original solution proposed by the author was incorrect/incomplete (graded 4/6). It is removed in this document.

(d) [4 points] Find the necessary and sufficient condition for $G_{m,n}$ to have an Eulerian circuit.

Answers:
For $G_{m,n}$ to have an Eulerian circuit, all vertices must have even degree, i.e., $(m + n - 2)$ is even and is an integer. Note that subtracting 2 does not change the parity of a number. Therefore, the necessary and sufficient condition is that $\boxed{m + n \text{ is even, i.e., } m \text{ and } n \text{ have the same parity}}$.

5. Graph Coloring

(a) [10 points] Recall that a proper region coloring is an assignment of a color to each region in a planar embedding of a graph such that adjacent regions receive different colors. Count the number of distinct ways of region coloring the graph D_4 (see below) with k colors for any positive integer k.

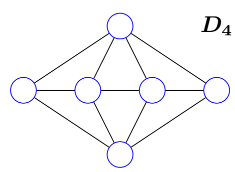

Figure 3: Graph D_4

Answers:
The original solution proposed by the author was incorrect/incomplete (graded 5/10). It is removed in this document.

(b) [8 points] Recall that a proper vertex coloring is an assignment of a color to each vertex in a graph such that adjacent vertices receive different colors. Count the number of distinct ways to vertex coloring the graph below with k colors for any positive integer k.

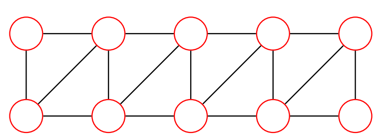

Figure 4: Graph for Question (5)(b)

Answers:
The original solution proposed by the author was incorrect/incomplete (graded 4/8). It is removed in this document.

6. Applications of Graph Theory

(a) [4 points] Seven people that assist to a conference want to have lunch together at a roundtable during the three days that the conference lasts. In order to get to know each other better, they decide to sit in such a way that two people are next to each other at most once. Can they achieve their goal? And what happens if the congress lasts 5 days?

Answers:

Represent each person by a vertex, and connect two vertices if the corresponding people sit next to each other. Then, each lunch arrangement corresponds to a C_7 in the graph. Also, each edge can appear in at most one C_7.

Note that K_7 has $\binom{7}{2} = 21$ edges, and each C_7 contains 7 edges. Therefore, at most $\lfloor \frac{21}{7} \rfloor = 3$ different lunch arrangements can be made such that two people are next to each other at most once.

If the congress lasts 3 days, $\boxed{\text{they can achieve their goal}}$.

However, if the congress lasts 5 days, $\boxed{\text{they cannot achieve their goal}}$.

(b) [8 points] At a dancing party, 10 participants form pairs to dance at each round. Suppose after many rounds, it is found that everyone has danced with exactly 2 other participants. How many different records could there be, if one is to record all pairs of participants who have danced? (Note that the records are not required to contain information on in which order have the participants danced.)

Answers:

Represent each participant by a vertex, connect two vertices if the corresponding participants have danced together. Then, we have a graph with 10 vertices, each of degree 2. We count the number of distinct graphs possible.

A graph where every vertex has degree 2 must consist of disjoint cycles. By Handshaking, $\sum \deg(v) = 2|E| \Rightarrow |E| = \frac{1}{2}(10 \cdot 2) = 10$.

We partition 10 vertices into disjoint cycles. Consider the possible cycle structures:

(1) One cycle C_{10}: There are $\frac{1}{2}(10-1)! = \frac{9!}{2}$ ways to arrange 10 vertices in a cycle.

(2) One C_9 and one C_1: Impossible since C_1 requires a self-loop, which is not allowed in simple graphs.

(3) One C_8 and one C_2: Impossible since C_2 is not a valid simple cycle.

(4) One C_7 and one C_3: $\binom{10}{7} \cdot \frac{1}{2}(7-1)! \cdot \frac{1}{2}(3-1)! = \binom{10}{7} \cdot \frac{6!}{2} \cdot \frac{2!}{2} = 120 \cdot 360 \cdot 1 = 43200$.

(5) One C_6 and one C_4: $\binom{10}{6} \cdot \frac{1}{2}(6-1)! \cdot \frac{1}{2}(4-1)! = \binom{10}{6} \cdot \frac{5!}{2} \cdot \frac{3!}{2} = 210 \cdot 60 \cdot 3 = 37800$.

(6) One C_5 and one C_5: $\frac{1}{2!}\binom{10}{5} \cdot \frac{1}{2}(5-1)! \cdot \frac{1}{2}(5-1)! = \frac{1}{2} \cdot 252 \cdot 12 \cdot 12 = 18144$.

(7) Two C_3 and one C_4: $\frac{1}{2!}\binom{10}{3}\binom{7}{3} \cdot \left(\frac{1}{2}(3-1)!\right)^2 \cdot \frac{1}{2}(4-1)! = \frac{1}{2} \cdot 120 \cdot 35 \cdot 1 \cdot 1 \cdot 3 = 6300$.

(8) One C_3, one C_3, one C_3, and one C_1: Impossible since C_1 is invalid.

Therefore, the total number of distinct records is

$$\boxed{\frac{9!}{2} + 43200 + 37800 + 18144 + 6300 = 181440 + 43200 + 37800 + 18144 + 6300 = 286884}.$$