COMP2121 Discrete Mathematics

25/26 Semester 1

Assignment 2

SHING, Zhan Ho Jacob 3036228892

1. Basics on Functions

(a) [4 points] For the function $h : \mathbb{Z} \to \mathbb{Z}$,

$$h(n) = \begin{cases} \frac{1}{3}n, & \text{if } n \text{ is divisible by 3,} \\ 3n, & \text{if } n \text{ is not divisible by 3,} \end{cases}$$

identify its (1) range, and whether it is (2) injective, (3) surjective, (4) bijective; if your answer is no, explain why.

Answers:

(1) **Case 1:** n is divisible by 3, i.e., $\forall k \in \mathbb{Z} : n = 3k$. Then, $h(n) = \frac{1}{3}(3k) = k$, which may or may not be divisible by 3.

Case 2: n is not divisible by 3 and $n = 3k + 1$ for some $k \in \mathbb{Z}$. Then, $h(n) = 3(3k + 1) = 9k + 3$, which is divisible by 3.

Case 3: n is not divisible by 3 and $n = 3k + 2$ for some $k \in \mathbb{Z}$. Then, $h(n) = 3(3k + 2) = 9k + 6$, which is also divisible by 3.

Thus, $h(n)$ is either divisible by 3 or not divisible by 3. $\boxed{\text{Therefore, the range of } h \text{ is } \mathbb{Z}.}$

(2) We can easily find a counterexample, e.g., $h(1) = h(9) = 3$. Thus, $\boxed{h \text{ is not injective}}$.

(3) By sub-problem (1), we have shown that the range of h is $\mathbb{Z}$, which is the same as the codomain of h. By definition, $\boxed{h \text{ is surjective}}$.

(4) Since h is not injective, by definition, $\boxed{h \text{ is not bijective}}$.

(b) [6 points] Functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are defined as follows. Determine by formulas the compositions $f \circ g$ and $g \circ g$.

$$f(x) = (x+2)^2 \quad \text{and} \quad g(x) = \begin{cases} 2x-6, & \text{if } x \geq 1, \\ -x-1, & \text{if } x < 1. \end{cases}$$

Answers:

(1) $\boldsymbol{f \circ g}$**:**

$$(f \circ g)(x) = \boxed{\begin{cases} (2x-4)^2, & \text{if } x \geq 1, \\ (-x+1)^2, & \text{if } x < 1. \end{cases}}$$

(2) $\boldsymbol{g \circ g}$**:**

Note that $g(x) \geq 1$ when $2x - 6 \geq 1 \Rightarrow x \geq \frac{7}{2} \geq 1$ and when $-x - 1 \geq 1 \Rightarrow x \leq -2$.

Also, $g(x) < 1$ when $2x - 6 < 1 \Rightarrow 1 \leq x < \frac{7}{2}$ and when $-x - 1 < 1 \Rightarrow 1 > x > -2$.

Thus, the piecewise function $g \circ g$ has cases: $x \in (-\infty, -2]$, $x \in (-2, 1)$, $x \in [1, \frac{7}{2})$, and $x \in [\frac{7}{2}, +\infty)$.

Case 1: $x \in (-\infty, -2]$. We have $g(x) \geq 1$, so $g(g(x)) = 2(-x-1) - 6 = -2x - 8$.

Case 2: $x \in (-2, 1)$, We have $g(x) < 1$, so $g(g(x)) = -(-x-1) - 1 = x$.

Case 3: $x \in [1, \frac{7}{2})$. We have $g(x) < 1$, so $g(g(x)) = -(2x-6) - 1 = -2x + 5$.

Case 4: $x \in [\frac{7}{2}, +\infty)$. We have $g(x) \geq 1$, so $g(g(x)) = 2(2x-6) - 6 = 4x - 18$.

Putting all cases together, we have

$$(g \circ g)(x) = \boxed{\begin{cases} -2x-8, & \text{if } x \leq -2, \\ x, & \text{if } -2 < x < 1, \\ -2x+5, & \text{if } 1 \leq x < \frac{7}{2}, \\ 4x-18, & \text{if } x \geq \frac{7}{2}. \end{cases}}$$

2. Relations and Functions

Prove or disprove the following statements.

(a) [4 points] For arbitrary function $f : \mathbb{Z} \to \mathbb{Z}$, the relation defined by

$$R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid f(x) - f(y) \text{ is even}\}$$

is reflexive and transitive.

Answers:

Reflexivity: Recall one of the properties of a function $f : A \to B$ defined by a relation R_0 is that $\forall x \in A, y_1, y_2 \in B : [(x\ R_0\ y_1) \wedge (x\ R_0\ y_2) \to (y_1 = y_2)]$. Therefore, for any pair (x, y) where $x, y \in \mathbb{Z}$ and $x = y$, we must have $f(x) = f(y) \Rightarrow f(x) - f(y) = 0$, which is even. Thus, $x\ R\ x$ for all $x \in \mathbb{Z}$. By definition, $\boxed{R \text{ is reflexive}}$.

Transitivity: Let $a, b, c \in \mathbb{Z}$ be arbitrary. Note that for $x\ R\ y$ to hold, $f(x)$ and $f(y)$ must either be both even or both odd. Also note that the image of $f : \mathbb{Z} \to \mathbb{Z}$ must be integers, which must either be even or odd. Then, when $a\ R\ b$ holds, we must have that the parity of $f(a)$ and $f(b)$ are the same. Similarly, when $b\ R\ c$ holds, we must have that the parity of $f(b)$ and $f(c)$ are the same. Thus, the parity of $f(a)$ and $f(c)$ must also be the same, which implies that $a\ R\ c$ holds. By definition, $\boxed{R \text{ is transitive}}$.

Conclusion: Since R is both reflexive and transitive, $\boxed{\text{the statement is true}}$.

(b) [4 points] For arbitrary function $f : \mathbb{Z} \to \mathbb{Z}$, the relation defined by

$$R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid f(x) \cdot f(y) \in \{-1, 0, 1\}\}$$

is symmetric and transitive.

Answers:

Observe that for $f(x) \cdot f(y) = 1$ to hold, $f(x)$ and $f(y)$ must both have the magnitude of 1 and the same sign. Similarly, for $f(x) \cdot f(y) = -1$ to hold, $f(x)$ and $f(y)$ must both have the magnitude of 1 but different signs. Finally, for $f(x) \cdot f(y) = 0$ to hold, at least one of $f(x)$ and $f(y)$ must be 0. That is, $f(x), f(y) \in \{-1, 0, 1\} \leftrightarrow f(x) \cdot f(y) \in \{-1, 1\}$, and $[(f(x) = 0) \vee (f(y) = 0)] \leftrightarrow f(x) \cdot f(y) = 0$.

Symmetry: We need to show that $x\ R\ y \to y\ R\ x$ holds for all $x, y \in \mathbb{Z}$. We try to prove by contradiction. Suppose $x\ R\ y$ holds but we have $y \not R\ x$. For the case when $f(x) \cdot f(y) \in \{-1, 1\}$, from $x\ R\ y$, we have $f(x), f(y) \in \{-1, 1\}$. Then, $f(y) \cdot f(x) \in \{-1, 1\}$, which implies that $y\ R\ x$ must hold. Then, for the case when $f(x) \cdot f(y) = 0$, from $x\ R\ y$, we have either or both of $f(x)$ and $f(y)$ equal to 0. Without loss of generality, suppose $f(x) = 0$. Then, $f(y) \cdot f(x) = f(y) \cdot 0 = 0$, which implies that $y\ R\ x$ must hold. Then, we have shown that in both cases, $y\ R\ x$ must hold. This contradicts our assumption that $y \not R\ x$. Therefore, $\boxed{R \text{ is symmetric}}$.

Transitivity: The transitivity can be easily disproved by a counterexample. Define $f_0 : \mathbb{Z} \to \mathbb{Z}$ as:

$$f_0(x) = \begin{cases} 5, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then, we have $f_0(-1) \cdot f_0(0) = 0 \in \{-1, 0, 1\}$ and $f_0(0) \cdot f_0(1) = 0 \in \{-1, 0, 1\}$, so $-1\ R\ 0$ and $0\ R\ 1$ hold. However, $f_0(-1) \cdot f_0(1) = 5 \cdot 5 = 25 \notin \{-1, 0, 1\}$, so $-1 \not R\ 1$. Therefore, $\boxed{R \text{ is not transitive}}$.

Conclusion: Since R is symmetric but not transitive, $\boxed{\text{the statement is false}}$.

3. Image of a Set

Let $f : A \to B$ be a function. Consider an arbitrary set $A' \subseteq A$. The image of set A' is defined to be the set

$$f(A') = \{y \in B \mid \exists x \in A' \text{ such that } y = f(x)\}.$$

(a) [4 points] Let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$, $A_1 = \{1, 2\}$, and $A_2 = \{2, 3\}$. Prove or disprove the statement: $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ holds for any function $f : A \to B$.

Answers:

Consider the function $f : A \to B$ defined as:

$$f(x) = \begin{cases} a, & x \in \{1, 3\}, \\ c, & x = 2. \end{cases}$$

Then, the image of $A_1 \cap A_2 = \{2\}$ is

$$f(A_1 \cap A_2) = f(\{2\}) = \{c\},$$

and the intersection of the images of A_1 and A_2 is

$$f(A_1) \cap f(A_2) = f(\{1, 2\}) \cap f(\{2, 3\}) = \{a, c\} \cap \{c, a\} = \{a, c\}.$$

Therefore, we have found a counterexample such that the statement does not hold. Thus, $\boxed{\text{the statement is false}}$.

(b) [6 points] Prove that for any subsets A_1, A_2 of A, $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ if f is injective.

Proof:

If f is injective, then for any $x_1, x_2 \in A$ where $x_1 \neq x_2$, we must have $f(x_1) \neq f(x_2)$. For any arbitrary set $A = \{a_1, a_2, \ldots\}$, we have $A' = f(A) = \{f(a_1), f(a_2), \ldots\}$ and $|A| = |A'|$, and for any $A_i \subseteq A$, we also have $A'_i = f(A_i) \subseteq A'$ and $|A_i| = |A'_i|$.

Case 1: $A_1 \cap A_2 = \varnothing$. Then, $f(A_1 \cap A_2) = f(\varnothing) = \varnothing$. Also, $f(A_1) \cap f(A_2) = A'_1 \cap A'_2 = \varnothing$ since $A_1 \cap A_2 = \varnothing$. Thus, $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ holds.

Case 2: $A_1 \cap A_2 \neq \varnothing$. Suppose A_1 and A_2 has some arbitrary common elements $c_1, c_2, \ldots, c_k$ where $k \in [1, \max(|A_1|, |A_2|)]$. Then, $f(A_1 \cap A_2) = \{f(c_1), f(c_2), \ldots, f(c_k)\}$. Also, suppose A_1 has some arbitrary unique elements $a_{11}, a_{12}, \ldots, a_{1m}$ where $m \in [0, |A_1| - k]$, and the same for A_2 with unique elements $a_{21}, a_{22}, \ldots, a_{2n}$ where $n \in [0, |A_2| - k]$. Then, $f(A_1) = \{f(c_1), f(c_2), \ldots, f(c_k), f(a_{11}), f(a_{12}), \ldots, f(a_{1m})\}$ and $f(A_2) = \{f(c_1), f(c_2), \ldots, f(c_k), f(a_{21}), f(a_{22}), \ldots, f(a_{2n})\}$. Thus, $f(A_1) \cap f(A_2) = \{f(c_1), f(c_2), \ldots, f(c_k)\} = f(A_1 \cap A_2)$. Therefore, $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ also holds.

Conclusion: Since $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ holds in both cases, $\boxed{\text{the statement is true}}$.

Q.E.D.

4. Asymptotics of Functions

(a) [8 points] For the following functions f, g, determine whether $f = O(g)$, $f = \Omega(g)$, or $f = \Theta(g)$. Justify your answers.

(1) $f(n) = 2n^2 + 3n + 1$ and $g(n) = n^2$.

Answers:
Note that $\forall n \geq 0$, we always trivially have $f(n) \geq g(n)$. That is, we can choose $c = 1$ and $n_0 = 0$, then $\forall n \geq n_0 : f(n) \geq cg(n)$ holds. This gives us $f \in \Omega(g)$.
Also note that $\forall n \geq 4$, we have $f(n) \leq 3g(n)$, Then, we can also choose $c = 3$ and $n_0 = 4$, then $\forall n \geq n_0 : f(n) \leq cg(n)$ holds. This gives us $f \in O(g)$.
Having shown that $f \in O(g)$ and $f \in \Omega(g)$, by definition, $\boxed{\text{we have } f \in \Theta(g)}$.

(2) $f(n) = \log_4 n$ and $g(n) = \log_2(\log_2 n)$.

Answers:
Note that we can transform $f(n)$ as

$$f(n) = \frac{\log_2 n}{\log_2 4} = \frac{1}{2} \log_2 n.$$

Then, we examine the growth rates of $f(n)$ and $g(n)$ as $n \to +\infty$:

$$\begin{aligned}
\lim_{n\to+\infty} \frac{f(n)}{g(n)} &= \lim_{n\to+\infty} \frac{\frac{1}{2}\log_2 n}{\log_2(\log_2 n)} \\
&= \frac{1}{2} \cdot \lim_{n\to+\infty} \frac{\log_2 n}{\log_2(\log_2 n)} \\
&= \frac{1}{2} \cdot \lim_{n\to+\infty} \frac{\frac{\ln n}{\ln 2}}{\frac{\ln(\log_2 n)}{\ln 2}} \\
&= \frac{1}{2} \cdot \lim_{n\to+\infty} \frac{\ln n}{\ln(\log_2 n)} \\
&= \frac{1}{2} \cdot \lim_{n\to+\infty} \frac{\ln n}{\ln(\frac{\ln n}{\ln 2})} \\
&= \frac{1}{2} \cdot \lim_{n\to+\infty} \frac{\ln n}{\ln(\ln n) - \ln(\ln 2)} \\
&= +\infty
\end{aligned}$$

Therefore, we have $f(n)$ grows asymptotically faster than $g(n)$, i.e., $f \in \Omega(g)$. We can verify this by choosing $c = 1$ and $n_0 = 16$. Then, $\forall n \geq n_0$, we have $f(n) \geq g(n)$. Thus, $\boxed{f \in \Omega(g)}$.

(3) $f(n) = n!$ and $g(n) = 100^n$. [Hint: Consider Stirling's formula.]

Answers:
By Stirling's formula, we have $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. Then, the limit evaluates as $\lim_{n\to\infty} \frac{f(n)}{g(n)} \approx \lim_{n\to\infty} \frac{\sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}}{100^n}$.
We can identify the dominant term being $n^{n+\frac{1}{2}}$, which grows faster than all other terms. Thus, the ratio $\frac{f(n)}{g(n)} \to +\infty$ as $n \to +\infty$.
Therefore, we have $f(n)$ grows asymptotically faster than $g(n)$, i.e., $\boxed{f \in \Omega(g)}$.

(b) [8 points] For any function $g : \mathbb{N} \to \mathbb{N}$, define $f_g : \mathbb{N} \to \mathbb{N}$ to be the function

$$f_g(n) = \underbrace{g \circ \cdots \circ g}_{n \text{ times}}(n-1).$$

Let g_1 be the function defined by $g_1(n) = 2n + 1$ for any $n \in \mathbb{N}$ and g_2 be the function defined by $g_2(n) = n^2$ for any $n \in \mathbb{N}$. Find a function $h : \mathbb{N} \to \mathbb{N}$ such that all the following requirements are satisfied:

- $h \in \Omega(f_{g_1})$ but $h \notin O(f_{g_1})$;
- $h \in O(f_{g_2})$ but $h \notin \Omega(f_{g_2})$.

Justify your answer.

Answers:
First, we observe the behaviour of f_{g_1}:

$$\begin{aligned}
f_{g_1}(0) &= 0, \\
f_{g_1}(1) &= g_1(n-1)\big|_{n=1} = 2n-1\big|_{n=1} = 1, \\
f_{g_1}(2) &= g_1(2n-1)\big|_{n=2} = 4n-1\big|_{n=2} = 7, \\
f_{g_1}(3) &= g_1(4n-1)\big|_{n=3} = 8n-1\big|_{n=3} = 23, \\
f_{g_1}(4) &= g_1(8n-1)\big|_{n=4} = 16n-1\big|_{n=4} = 63, \\
&\cdots
\end{aligned}$$

We can see the coefficient of the dominant term n is 2^n. Also, in each iteration, the coefficient is multiplied by n and some abitrary number is added. Then, we can identify the dominant term of $f_{g_1}(n)$ being $2^n n$.
Similarly, we observe the behaviour of f_{g_2}:

$$\begin{aligned}
f_{g_2}(0) &= 0, \\
f_{g_2}(1) &= g_2(n-1)\big|_{n=1} = (n-1)^2\big|_{n=1} = 0, \\
f_{g_2}(2) &= g_2((n-1)^2)\big|_{n=2} = (n-1)^4\big|_{n=2} = 1, \\
f_{g_2}(3) &= g_2((n-1)^4)\big|_{n=3} = (n-1)^8\big|_{n=3} = 256, \\
f_{g_2}(4) &= g_2((n-1)^8)\big|_{n=4} = (n-1)^{16}\big|_{n=4} = 43,046,721, \\
&\cdots
\end{aligned}$$

The exponent of the dominant term n is 2^n. Then, the dominant term of $f_{g_2}(n)$ is n^{2^n}.
We would like to find a function $h : \mathbb{N} \to \mathbb{N}$ such that at large n, h grows asymptotically faster than f_{g_1} but slower than f_{g_2}. We choose $h(n) = n^n$.
Consider the ratio $\frac{h(n)}{f_{g_1}(n)} = \frac{n^n}{2^n n} = \frac{n^{n-1}}{2^n}$ as $n \to +\infty$. It is clear that n^{n-1} grows much faster than 2^n. Thus, $\lim_{n\to\infty} \frac{h(n)}{f_{g_1}(n)} = +\infty$, which implies that $h \in \Omega(f_{g_1})$ but $h \notin O(f_{g_1})$.
Now, consider the ratio $\frac{h(n)}{f_{g_2}(n)} = \frac{n^n}{n^{2^n}} = \frac{1}{n^{2^n - n}}$ as $n \to +\infty$. It is trivial to show that $\lim_{n\to\infty} \frac{h(n)}{f_{g_2}(n)} = 0$, which implies that $h \in O(f_{g_2})$ but $h \notin \Omega(f_{g_2})$.
Therefore, we have found a function $\boxed{h(n) = n^n}$ that satisfies all the requirements.

5. Basic Counting

(a) [4 points] Among all positive numbers that devide 4050 exactly, how many are multiples of 45?

> **Answers:**
> Observe the prime factorisation of $4050 = 2 \cdot 3^4 \cdot 5^2$. Also note the fact that an integer $d \leq n$ divides n exactly iff every prime factor of d is also a prime factor of n, i.e., the multiples of at least one prime factor of n divide n exactly.
>
> Then, the divisors of 4050 must have the form $2^a \cdot 3^b \cdot 5^c$ where $a \in [0,1] \cap \mathbb{Z}$, $b \in [0,4] \cap \mathbb{Z}$, and $c \in [0,2] \cap \mathbb{Z}$. We have 2 ways to choose a, 5 ways to choose b, and 3 ways to choose c. Thus, the total number of divisors of 4050 is $2 \cdot 5 \cdot 3 = 30$.
>
> Now, we denote S_n as the set of multiples of n with $\forall x \in S_n : 1 \leq x \leq 4050$. Then, $|S_{45}| = \left\lfloor \frac{4050}{45} \right\rfloor = 90$. Note that every multiple of 45 must have the form $C \cdot 3^2 \cdot 5$, where $C \in \mathbb{Z}^+$ such that $C \cdot 3^2 \cdot 5 \leq 4050$. Then, for the multiples of 45 to also be divisors of 4050, C must be in the form $2^a \cdot 3^b \cdot 5^c$ where $a \in [0,1] \cap \mathbb{Z}$, $b \in [0,2] \cap \mathbb{Z}$, and $c \in [0,1] \cap \mathbb{Z}$. We have 2 ways to choose a, 3 ways to choose b, and 2 ways to choose c. Thus, the total number of multiples of 45 that are also divisors of 4050 is $2 \cdot 3 \cdot 2 = 12$.
>
> Therefore, $\boxed{\text{there are 12 such numbers}}$.

(b) [4 points] There are 2 identical red balls and 3 identical black balls. You are going to put them into 5 different boxes. If each box can contain at most 2 balls, how many ways are there to put the balls in the boxes?

> **Answers:**
> Consider the distribution of balls, we can solve the problem by cases.
>
> **Case 1:** Every box contains exactly 1 ball. This is the same as the number of 5-permutations of 2 red balls and 3 black balls, which is $\dfrac{5!}{2!3!} = 10$.
>
> **Case 2:** One box contains 2 balls, another box contains no balls, and the remaining 3 boxes contain 1 ball each. We can choose 1 box out of 5 to contain 2 balls (label it as B_2), and then choose 1 box out of 4 to contain no balls (B_0). The remaining 3 boxes will contain 1 ball each (B_{1i}, $i = 1, 2, 3$). There are $5 \cdot 4 = 20$ ways to choose the boxes. In each of these choices, B_2 can either contain 2 red balls, 2 black balls, or 1 of each colour. For the two red balls case, B_{1i} must contain 1 black ball each, which is 1 way. For the two black balls case, we have 3 ways to choose one of B_{1i} to contain the remaining black ball, and the other two boxes will contain 1 red ball each. For the 1 red and 1 black balls case, we have 3 ways to choose one of B_{1i} to contain the remaining red ball, and the other two boxes will contain 1 black ball each. Thus, there are $1 + 3 + 3 = 7$ ways to distribute the balls in each choice of boxes. Therefore, there are $20 \cdot 7 = 140$ ways in this case.
>
> **Case 3:** Two boxes contain 2 balls each, and one box contains 1 ball, and the remaining 2 boxes contain no balls. We first choose 1 box to hold 1 ball (B_1), there are 5 ways. B_1 can either hold a black ball or a red ball (2 ways). If B_1 holds a black ball, then we either choose 1 box out of 4 to hold 2 red balls (4 ways) and 1 box out of 3 to hold 2 black balls (3 ways), or we choose 2 boxes out of 4, each holding the same colour (6 ways) This gives us $4 \cdot 3 + 6 = 18$ ways. If B_1 holds a red ball, then we must choose 1 box out of 4 to hold 2 black balls (4 ways) and 1 box out of 3 to hold 1 red and 1 black balls (3 ways). This gives us $4 \cdot 3 = 12$ ways. Therefore, there are $5 \cdot (18 + 12) = 150$ ways in this case.
>
> Putting all cases together, we have $\boxed{10 + 140 + 150 = 300 \text{ ways in total}}$.

(c) [4 points] In a game, the player needs to move from the point $(0, 0)$ to the point (a, b) where $a \geq b > 0$ are integers. At each point (x, y), the player can either move right to $(x + 1, y)$ or move up to $(x, y + 1)$. It is forbidden to move up for two successive times. How many different ways are there for the player to reach (a, b)?

> **Answers:**
> Note that since the player can move only one unit to the right or upwards, regardless of the path the player chooses, exactly $(a + b)$ steps are required to reach (a, b) from $(0, 0)$, with a steps to reach the line $x = a$ and b steps to reach the line $y = b$.
>
> The path is a $(a + b)$-sequence of the set $\{U, R\}$, where U denotes moving upwards and R denotes moving to the right. In addition, we must have no pairs of U's that are adjacent to each other. The sequence must have a R's and b U's.
>
> Consider a sequence of a R's and $(a + 1)$ gaps between each pair of R's, including the head and the tail. We need to choose b gaps to put the b U's, that is $\binom{a+1}{b}$ ways. Thus, $\boxed{\text{there are } \binom{a+1}{b} \text{ ways in total}}$.

6. Counting Sets and Relations

(a) [4 points] In a group of 15 computer scientists, each scientist has collaborated with at least 8 other members of the group. Prove that there exist 3 members, such that each one of them has collaborated with the other two members.

Proof:

Let A be the set of the 15 computer scientists, $x \; R \; y$: "x has collaborated with y". Note that R is symmetric. Also define the set $C_x = \{y \in A - \{x\} \mid x \; R \; y\}$ as the set of collaborators of x. Then, we have $\forall x \in A : |C_x| \geq 8$.

Assume, for contradiction, that no 3 members have mutually collaborated.

For any arbitrary $x \in A$, and for some arbitrary $y \in C_x$, since x and y have collaborated, we also have $x \in C_y$. Since no 3 members have mutually collaborated, we have $\nexists z : (z \in C_x \wedge z \in C_y)$, where x, y, z are distinct.

Note that $|C_x - \{y\}| \geq 7$ and $\left|C_y - \{x\}\right| \geq 7$. Since $C_x - \{y\}$ and $C_y - \{x\}$ are disjoint, we have $|C_x - \{y\}| + \left|C_y - \{x\}\right| \geq 14$. However, since $A - \{x, y\}$ has only 13 members, and $(C_x - \{y\}) \cup (C_y - \{x\}) \subseteq A - \{x, y\}$, we must have $\left|(C_x - \{y\}) \cup (C_y - \{x\})\right| \leq 13$. This contradicts the fact that $|C_x - \{y\}| + \left|C_y - \{x\}\right| \geq 14$.

Therefore, our assumption is false, and $\boxed{\text{there exist 3 members who have mutually collaborated}}$.

Q.E.D.

(b) [6 points] Let A, B, C be sets with $|A| = |B| = |C| = n$ and $|A \cup B \cup C| = m$. Find the minimal value of $|A \cap B \cap C|$.

Answers:

Recall that $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$, substitue the given values, we have

$$m = 3n - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$
$$|A \cap B \cap C| = m - 3n + |A \cap B| + |A \cap C| + |B \cap C| \tag{1}$$

Since $|A \cup B| = |A| + |B| - |A \cap B|$, and $|A \cup B| \leq |A \cup B \cup C| = m$, we have

$$\begin{aligned} |A| + |B| - |A \cap B| &\leq m \\ 2n - |A \cap B| &\leq m \\ |A \cap B| &\geq 2n - m \end{aligned}$$

Similarly, we find the lower bounds for $|A \cap C|$ and $|B \cap C|$ to also be $2n - m$.

Then, equation (1) becomes

$$\begin{aligned} |A \cup B \cup C| &\geq m - 3n + 3(2n - m) \\ &= 3n - 2m \end{aligned}$$

Moreover, since $|A \cap B \cap C| \geq 0$, we have that $\min |A \cup B \cup C| = \boxed{\max(0, 3n - 2m)}$.

(c) [6 points] Let A be a set with $|A| = 5$. Count the number of distinct (i) symmetric relations and (ii) asymmetric relations that can be constructed on A. An asymmetric relation R is a relation such that, for every pair of distinct x and y, $x\ R\ y \Rightarrow \neg(y\ R\ x)$.

Answers:

Express a relation on A as a 5-by-5 matrix, where each entry $r_{ij} = \begin{cases} 1, & \text{if } x_i\ R\ x_j, \\ 0, & \text{if } x_i \not R\ x_j \end{cases}$, and $x \in A$.

(i) **Symmetric Relations:**

If R is symmetric, then its matrix representation must also be a symmetric matrix. Since the main diagonal of a symmetric matrix is arbitrary, for each entry, we can freely choose between 1 and 0. Then, we have $2^5 = 32$ choices. For the off-diagonals, we must have $r_{ij} = r_{ji}$, where $i \neq j$. Then, for r_{ij} where $i \in [2,5]$ and $j \in [1, i-1]$, i.e., the lower triangular part of the matrix excluding the main diagonal, we can freely choose between 1 and 0, and the corresponding r_{ji} is constrained. We have $2^4 \cdot 2^3 \cdot 2^2 \cdot 2^1 = 1024$ choices.

Therefore, in total, there are $32 \cdot 1024 = \boxed{32,768}$ distinct symmetric relations.

(ii) **Asymmetric Relations:**

Note that the requirement of asymmetry implies that whenever $r_{ij} = 1$, its corresponding r_{ji} must be 0, and vice versa, but if $r_{ij} = r_{ji} = 0$, the requirement is still satisfied. Also, the main diagonal must be all 0's as asymmetry requires irreflexivity. Then, for each of the off-diagonal pairs (r_{ij}, r_{ji}) where $i \in [2,5]$ and $j \in [1, i-1]$, there are 3 possibilities: $(1,0)$, $(0,1)$, and $(0,0)$. Note that there are $4 + 3 + 2 + 1 = 10$ such pairs. Thus, we have $3^{10} = 59,049$ choices.

Therefore, in total, there are $\boxed{59,049}$ distinct asymmetric relations.

7. Pigeonhole Principle

(a) [4 points] Show that if you have a set A of 5 points $(x_i, y_i) \in \mathbb{Z}^2$, there must exist distinct j, k such that, for $(x_j, y_j), (x_k, y_k) \in A$, the midpoint $\left(\frac{x_j + x_k}{2}, \frac{y_j + y_k}{2}\right)$ also lies in $\mathbb{Z}^2$.

Proof:

For the midpoint of $(x_j, y_j), (x_k, y_k) \in A$ to lie in $\mathbb{Z}^2$, we require both the of their x- and y-coordinates to have the same parity, such that the sums of their x- and y-coordinates are even, hence divisible by 2. There can be 4 possible parity cases, namely $\{(\text{even}, \text{even}), (\text{even}, \text{odd}), (\text{odd}, \text{even}), (\text{odd}, \text{odd})\}$. We have 5 points distributed among the 4 cases, by the Pigeonhole Principle, at least two of them are in the same parity case. Therefore, there are at least 2 distinct points in A whose sums of x- and y-coordinates are of the same parity, and their midpoint lies in $\mathbb{Z}^2$.

We have shown that $\boxed{\text{such distinct } j, k \text{ exist}}$.

Q.E.D.

(b) [4 points] For any $n \in \mathbb{N}$, prove that for any collection of $n + 1$ distinct numbers drawn from $\{1, 2, \ldots, 2n\}$, there must be at least one pair of numbers such that one of them is a multiple of the other.

Proof:

Note that every positive integer x can be written in the form of $x = 2^k \cdot m$, where m is odd and $k \geq 0$. Then, each $x \in \{1, 2, \ldots, 2n\}$ corresponds to a unique odd factor $m \in \{1, 3, 5, \ldots, 2n-1\}$. There are n such odd factors in total. Since we are drawing $n + 1$ distinct numbers, by the Pigeonhole Principle, at least two of them must correspond to the same odd factor. Let $a = 2^s \cdot m$ and $b = 2^t \cdot m$ be the two numbers, where $s, t \geq 0$. Without loss of generality, assume $s < t$, then we have

$$\frac{b}{a} = \frac{2^t \cdot m}{2^s \cdot m} = 2^{t-s},$$

which is an integer. Thus, b is a multiple of a.

We have shown that $\boxed{\text{such a pair of numbers exist}}$.

Q.E.D.

8. Conditional Probability

You are training a classifier for spam detection using machine learning. Each data point (x, y) in the dataset represents an email, which can be categorized as either spam or non-spam, with the content of the email denoted as x and a label $y \in$ spam, non-spam.

(a) [4 points] Given that 10% of the data in the dataset is labelled as spam and 90% is labelled as non-spam, and that the model's output satisfies the following conditions:

$$P(\text{output} = \text{spam} \mid \text{label} = \text{spam}) = 0.8, \quad P(\text{output} = \text{non-spam} \mid \text{label} = \text{non-spam}) = 0.9,$$

determine the probability that the model's output matches the true label when the input is randomly selected from the dataset.

Answers:
Let L be the event that the mail is labelled as spam, $\overline{L}$ be the event that the mail is labelled as non-spam, O be the event that the model outputs spam for the mail, and $\overline{O}$ be the event that the model outputs non-spam for the mail.
From the given information, we have

$$P(L) = 0.1, \quad P(\overline{L}) = 0.9,$$
$$P(O \mid L) = 0.8, \quad P(\overline{O} \mid \overline{L}) = 0.9.$$

We need to find $P(L \cap O) + P(\overline{L} \cap \overline{O})$.
By conditional probabilities, we have $P(O \mid L) = \frac{P(L \cap O)}{P(L)} \Leftrightarrow P(L \cap O) = P(O \mid L) \cdot P(L) = 0.8 \cdot 0.1 = 0.08$. Similarly, we have $P(\overline{O} \cap \overline{L}) = P(\overline{O} \mid \overline{L}) \cdot P(\overline{L}) = 0.9 \cdot 0.9 = 0.81$.
Therefore, the required probability is $P(\text{output matches label}) = 0.08 + 0.81 = \boxed{0.89}$.

(b) [4 points] Continuing from part (a), calculate the probability that the true label is spam given that the model predicts spam.

Answers:
We need to find $P(L \mid O) = \frac{P(L \cap O)}{P(O)}$.
By the law of total probability, we have $P(O) = P(O \mid L)P(L) + P(O \mid \overline{L})P(\overline{L}) = 0.8 \cdot 0.1 + \left[1 - P(\overline{O} \mid \overline{L})\right] \cdot 0.9 = 0.08 + (1 - 0.9) \cdot 0.9 = 0.08 + 0.1 \cdot 0.9 = 0.08 + 0.09 = 0.17$.
Therefore, $P(L \mid O) = 0.08/0.17 = \boxed{\frac{8}{17}}$.

(c) [4 points] Suppose the labels are manually provided by users, introducing noise in the dataset. This means some emails have incorrect labels (i.e., the label has been flipped from spam to non-spam or vice versa). With the same conditions as in part (a), and given that 5% of the emails have incorrect labels, determine the probability that the model categorizes a random email from the dataset correctly, regardless of whether the label is accurate.

Answers:
Let C be the event that the label is correct, and $\overline{C}$ be the event that the label is flipped. We have $P(C) = 0.95$ and $P(\overline{C}) = 0.05$.
When the label is correct (C), from part (a), we know the probability that the model's output matches the label is 0.89. Since the label is correct, this is also the probability that the model categorizes the email correctly.
When the label is flipped ($\overline{C}$), the model's output matches the (incorrect) label with the same probability 0.89. Therefore, the model categorizes the email correctly (i.e., matches the true label) with probability $1 - 0.89 = 0.11$.
By the law of total probability, the probability that the model categorizes a random email correctly is

$$\begin{aligned} P(\text{correct}) &= P(\text{correct} \mid C)P(C) + P(\text{correct} \mid \overline{C})P(\overline{C}) \\ &= 0.89 \cdot 0.95 + 0.11 \cdot 0.05 \\ &= 0.8455 + 0.0055 \\ &= \boxed{0.851}. \end{aligned}$$

(d) [4 points] There are n passengers queuing to board a bus one by one. Each passenger has an assigned seat. However, the first passenger does not follow the seating arrangement and chooses a seat at random. Subsequently, each passenger takes their assigned seat if it is available; otherwise, they select a seat uniformly at random from the remaining unoccupied seats. Determine the probability that the last passenger sits in the assigned seat.

Answers:

Note that when there is only 1 passenger, they will always sit in their assigned seat. Therefore, the probability is 1.

For $n \geq 2$, denote E_n^m as the event that the n-th passenger sits in the seat numbered m. We are required to find $P(E_n^n)$.

Note that there are three cases:

- **Case 1 – E_1^1**: The first passenger chooses seat 1. In this case, all subsequent passengers will find their assigned seats unoccupied and sit in them. Thus, $P(E_n^n \mid E_1^1) = 1$.
- **Case 2 – E_1^n**: The first passenger chooses seat n. In this case, the last passenger will find their assigned seat occupied and must choose to sit somewhere else. Thus, $P(E_n^n \mid E_1^n) = 0$.
- **Case 3 – E_1^x, where $x \in \{2, 3, 4, \ldots, n-1\}$**: $P(E_1^x) = \frac{n-2}{n}$. Now, assume the 1st passenger chooses some seat, say seat x_0. Then, passengers $2, 3, \ldots, x_0 - 1$ will find their assigned seats unoccupied and sit in them. When passenger x_0 boards the bus, the problem reduces to as if the x_0-th passenger were the first to board the bus with $n - x_0 + 1$ seats remaining. The passenger, can either sit in seat 1, allowing all subsequent passengers to sit in their assigned seats, or sit in seat n, so that the last passenger cannot sit in their assigned seat, or sit in some other seat, until the problem reduces to only two passengers remaining. That is, the $(n-1)$-th passenger either sits in seat 1 or seat n. Therefore, we have $P(E_n^n \mid E_1^x) = P(E_{n-x+1}^{n-x+1}) = \frac{1}{2}$.

Therefore, by the law of total probability, we have

$$\begin{aligned} P(E_n^n) &= P(E_n^n \mid E_1^1)P(E_1^1) + P(E_n^n \mid E_1^n)P(E_1^n) + P(E_n^n \mid E_1^x)P(E_1^x) \\ &= 1 \cdot \frac{1}{n} + 0 \cdot \frac{1}{n} + \frac{1}{2} \cdot \frac{n-2}{n} \\ &= \frac{1}{n} + \frac{n-2}{2n} \\ &= \frac{n}{2n} \\ &= \frac{1}{2}. \end{aligned}$$

Thus, we have $\boxed{P(E_n^n) = \begin{cases} \frac{1}{2} & \text{if } n \geq 2 \\ 1 & \text{if } n = 1 \end{cases}}$.

(e) [4 points] A family has two children. It is known that at least one of them is a girl born on a Monday. What is the probability that both children are girls?

Answers:

Let $G = \{M, F\}$ be the set of genders, and $D = \{\text{Mon}, \text{Tue}, \text{Wed}, \text{Thu}, \text{Fri}, \text{Sat}, \text{Sun}\}$ be the set of days in a week. Then, the sample space S is given by $S = (G \times D) \times (G \times D)$ and $|S| = 2 \cdot 7 \cdot 2 \cdot 7 = 196$.

Denote A as the event that at least one girl was born on a Monday, and B as the event that both children are girls. We are required to find $P(B \mid A)$.

To find $|A|$, we consider the complement $\overline{A}$, which is that there is no girl, or if there is a girl, she was not born on a Monday. For two children, there are $13 \cdot 13 = 169$ such outcomes in total. Thus, $|A| = |S| - \left|\overline{A}\right| = 196 - 169 = 27$.

To find $|A \cap B|$, first consider B, whose cardinality is $|B| = 7 \cdot 7 = 49$. Then, count the number of outcomes in B that none of the girls was born on a Monday. There are $6 \cdot 6 = 36$ such outcomes. Thus, $|A \cap B| = |B| - 36 = 49 - 36 = 13$.

Therefore, by conditional probability, we have

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \boxed{\frac{13}{27}}.$$